ON THE DESIGN OF OVERSAMPLED FILTER BANKS FOR CHANNEL CODING

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ABSTRACT

Oversampled filter banks have been considered for channel coding, because they introduce redundancy into the subband representation of signals and permit more freedom in their design than critically sampled structures. In this paper, based on the knowledge of the channel noise's covariance matrix, we propose a constrained design for the synthesis filter bank in order to minimise the noise power in the decoded signal, subject to admitting perfect reconstruction. For the special case of paraunitary filter banks, a suboptimal iterative design is presented, which highlights the potential benefits of this approach, as demonstrated by a design example.

1. INTRODUCTION

The redundancy and design freedom afforded by oversampled filter banks (OSFBs) has in the past been exploited for robustness towards quantisation of subband signals [1, 2, 3], reconstruction of erased or erroneous subband samples [4], or for the design of error correction codecs [5, 6]. In the initial days of OSFBs, the general redundancy in the subband domain was used to attain a robust data representation [1]. In [3], the filter bank characteristics are specifically geared towards the spectral shaping of the quantisation noise in the subband domain. More recently, for a given analysis filter bank, in [5] the design freedom in selecting a synthesis filter bank is utilised by projecting away from the noise space, which is assumed to have a reduced rank.

This paper analyses the application of OSFBs as potential channel coders. Based on a brief description of filter banks in Sec. 2, the channel coding structure is presented in Sec. 3. With the aim of minimising the impact of additive channel noise on the decoded signal, we derive a noise power term similar to [3], which can be utilised as a cost function for the channel coder design. We propose a constrained optimisation scheme for the synthesis filter bank in Sec. 4, which aims to minimise the channel noise power at the decoder output subject to the filter bank being perfectly reconstructing. The appeal of such a channel coder design is motivated by results obtained by a suboptimal approach, which is demonstrated by an example in Sec. 5. The approach is discussed in Sec. 6 and conclusions are drawn in Sec. 7.

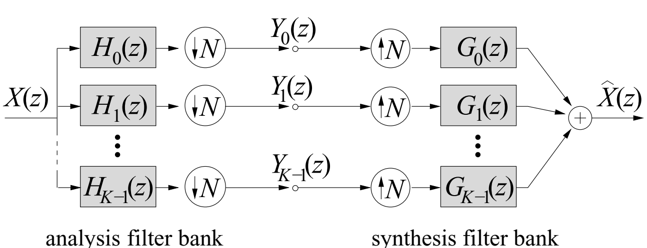

Figure 1: Subband decomposition of a signal $X(z)$.

2. FILTER BANKS

Based on the description of basic filter bank structures in Sec. 2.1, a transmission model is discussed in Sec. 3 with respect to coding, which will enable us to later formulate a design criterion.

2.1 Oversampled Filter Banks

Fig. 1 shows a general filter bank structure comprising an analysis and a synthesis stage. The analysis filter bank splits a fullband signal $X(z)$ into K frequency bands by a series of bandpass filters $H_k(z)$, $k = 0(1)K - 1$, and decimates by a factor $N \leq K$, resulting in so-called "subband" signals $Y_k(z)$. The dual operation of reconstructing a fullband signal from the K subband signals is accomplished by a synthesis filter bank, where upsampling by N is followed by interpolation filters $G_k(z)$, $k = 0(1)K - 1$.

The purpose of oversampling by a ratio $K/N > 1$ rather than a critical decimation by K has application specific reasons. Classically, filter banks comprise of a series of bandpass filters. Non-critical decimation of the resulting subbands will permit the benefit of lower computational complexity while avoiding aliasing in the subbands, which would otherwise limit the performance of, for example, adaptive filters operating independently on the various subband signals [7]. When processing the subband signals $Y_k(z)$ in Fig. 1 independently, the information that is located in the overlap regions of adjacent bandpass filters has to be made fully available to the at least two subbands sharing this spectral region. Unless overlap regions are avoided by permitting band gaps in the overall transfer function of Fig. 1, it is therefore intuitively clear that oversampling is required, and the redundancy must be placed at frequencies where the analysis filters spectrally overlap.

The redundancy afforded by OSFBs has more recently attracted attention for channel coding [5, 6]. There, a coding rate $N/K < 1$ can ensure robustness against noise interference, with the aim of restoring noise corrupted samples due to the redundant format in which the data is transmitted. The analysis and synthesis filter banks function as encoder and decoder, while the filters $H_k(z)$ and $G_k(z)$ are no longer limited to a bandpass design, but will rather be selected according to the characteristics of the interfering noise.

2.2 Polyphase Matrices

For implementation and analysis purposes, OSFBs as shown in Fig. 1 are conveniently represented by polyphase analysis and synthesis matrices. The former is based on a polyphase expansion of the analysis filters $H_k(z)$

$$H_k(z) = \sum_{n=0}^{N-1} z^{-n} H_{k,n}(z^N) \quad , \tag{1}$$

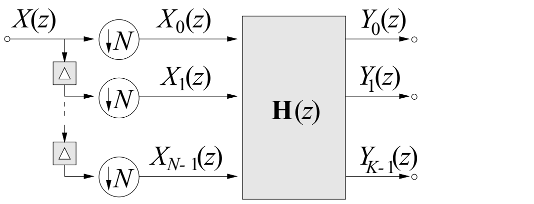

Figure 2: Analysis filter bank with demultiplexer and $\mathbf{H}(z)$ describing an $N \times K$ convolutive MIMO system.

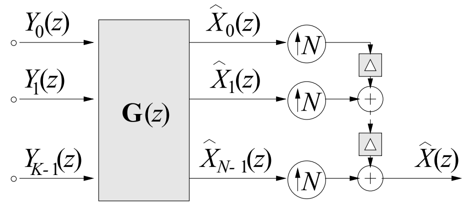

Figure 3: Synthesis filter bank with $\mathbf{G}(z)$ describing a $K{\times}N$ MIMO system followed by a multiplexer.

with polyphase components $H_{k,n}(z)$, and a similar decomposition of the input signal $X(z)$

$$X(z) = \sum_{n=0}^{N-1} z^{-N+n-1} X_n(z^N) \quad . \tag{2}$$

with polyphase components $X_n(z)$. This allows to denote the subband signals as

$$\begin{bmatrix} Y_0(z) \\ \vdots \\ Y_{K-1}(z) \end{bmatrix} = \underbrace{\begin{bmatrix} H_{0,0}(z) & \dots & H_{0,N-1}(z) \\ \vdots & \ddots & \vdots \\ H_{K-1,0}(z) & \dots & H_{K-1,N-1}(z) \end{bmatrix}}_{\mathbf{H}(z)} \begin{bmatrix} X_0(z) \\ \vdots \\ X_{N-1}(z) \end{bmatrix} \tag{3}$$

where the polyphase analysis matrix $\mathbf{H}(z) \in \mathbb{C}^{K\times N}(z)$ enables to structure the analysis filter bank as given in Fig. 2.

Analogously, a polyphase synthesis matrix $\mathbf{G}(z) \in \mathbb{C}^{N\times K}(z)$ can be defined based on a polyphase expansion of $G_k(z)$, yielding the flow graph shown in Fig. 3.

A filter bank system is perfectly reconstructing if

$$\mathbf{G}(z)\mathbf{H}(z) = z^{-\Delta}\mathbf{I}_N \quad . \tag{4}$$

The design of such a system can be demanding in terms of the number of coefficients that need to be optimised. A reduction of the parameter space by, for example, deriving all K filters from a prototype by modulation [2, 8] or by permitting only symmetric filter impulse responses [8, 4] often makes the problem tractable.

3. CHANNEL CODING VIA FILTER BANKS

The model of a channel codec based on filter banks is given in Fig. 4 similar to [5]. In general, the analysis filter bank is given, and should form a perfect reconstruction system with the synthesis bank [6]. For OSFBs, there is no unique $\mathbf{G}(z)$ that inverts $\mathbf{H}(z)$, given that $\mathbf{H}(z)$ admits a left inversion. Amongst the manifold of possible solutions, the design freedom for choosing $\mathbf{G}(z)$ is exploited such that the impact of noise on the transmitted signal is minimised in some sense.

The polyphase components $\hat{X}_n(z)$ of the received signal in Fig. 4 can be collected in a vector $\underline{\hat{X}}(z)$, which is given by

$$\underline{\hat{X}}(z) = \mathbf{G}(z)\left(\underline{Y}(z) + \underline{W}(z)\right) \tag{5}$$

whereby $\underline{Y}(z) = \mathbf{H}(z)\underline{X}(z) \in \mathbb{C}^K(z)$ and $\underline{W}(z) \in \mathbb{C}^K(z)$ contain the subband signal components of the transmitted data and the noise, respectively. Selecting perfect reconstruction filter banks according to (4), and w.l.o.g. setting $\Delta = 0$, an error vector

$$\underline{E}(z) = \underline{X}(z) - \underline{\hat{X}}(z) = -\mathbf{G}(z)\underline{W}(z) \tag{6}$$

is obtained.

To assess the total received noise variance σ_e^2, let the N-element vector $\mathbf{e}[m]$ contain the N time series with time index m associated with the z-domain quantities in $\underline{E}(z)$. Thus,

$$\sigma_e^2 = \frac{1}{N}\mathrm{tr}\{\mathcal{E}\left\{\mathbf{e}[m]\,\mathbf{e}^{\mathrm{H}}[m]\right\}\} \quad , \tag{7}$$

whereby $\mathrm{tr}\{\cdot\}$ denotes trace and $\mathcal{E}\{\cdot\}$ is the expectation operator. With $\mathbf{g}_{n,k}$ containing the coefficients of the nth polyphase component of the kth synthesis filter such that $G_{n,k}(z) = [1 \;\; z^{-1} \;\; \dots \;\; z^{-L_g/N+1}] \cdot \mathbf{g}_{n,k}$, and L_g being the length of the synthesis filters $G_k(z)$, the noise component at the decoder output can be formulated as

$$\begin{aligned} \mathbf{e}[m] &= \begin{bmatrix} \mathbf{g}_{0,0}^{\mathrm{T}} & \mathbf{g}_{0,1}^{\mathrm{T}} & \dots & \mathbf{g}_{0,K-1}^{\mathrm{T}} \\ \vdots & \ddots & & \vdots \\ \mathbf{g}_{N-1,0}^{\mathrm{T}} & \mathbf{g}_{N-1,1}^{\mathrm{T}} & \dots & \mathbf{g}_{N-1,K-1}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{w}_0[m] \\ \vdots \\ \mathbf{w}_{K-1}[m] \end{bmatrix} \\ &= \mathbf{G}\;\mathbf{w}[m] \end{aligned} \tag{8}$$

with $\mathbf{G} \in \mathbb{C}^{N\times KL_p/N}$ holding the coefficients of $\mathbf{G}(z)$, and $\mathbf{w}[m] \in \mathbb{C}^{KL_p/N}$. In (8), each vector $\mathbf{w}_k[m]$ contains L_g tap delay line values of the time series associated with the noise quantity $W_k(z)$, $k = 0(1)K-1$, in Fig. 4. Substituting (8) into (7), we obtain similar to [3] for the received noise variance

$$\begin{aligned} \sigma_e^2 &= \frac{1}{N}\,\mathrm{tr}\left\{\mathbf{G}\mathbf{R}_{ww}\mathbf{G}\right\} & (9) \\ &= \frac{1}{N}\,\mathrm{tr}\left\{\mathbf{G}\mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{\mathrm{H}}\mathbf{G}^{\mathrm{H}}\right\} \quad , & (10) \end{aligned}$$

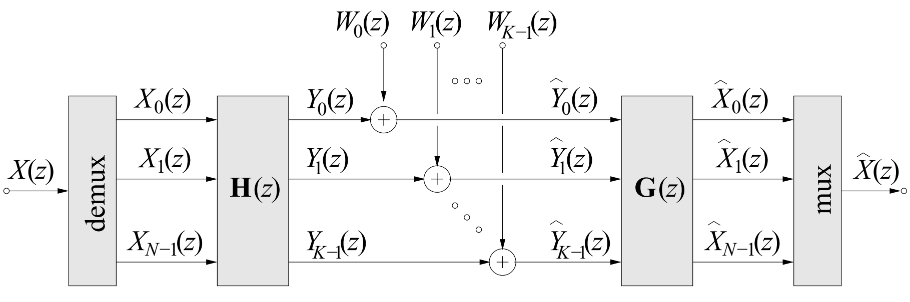

Figure 4: General setup of codec consisting of analysis filter banks as encoder, an additive model for noise interference, and a synthesis filter bank as decoder.

where $\mathbf{R}_{ww} = \mathcal{E}\{\mathbf{w}_n\mathbf{w}_n^{\mathrm{H}}\}$ with modal matrix $\mathbf{Q}$ and

$$\mathbf{\Lambda} = \mathrm{diag}\{\lambda_0,\ \lambda_1\ \ldots\ \lambda_{KL_p/N-1}\} \qquad (11)$$

with eigenvalues $\lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_{KL_p/N-1}$ arranged in ascending order for later convenience. Therefore, ideally the rows of $\mathbf{G}$ should lie in a subspace spanned by those eigenvector in $\mathbf{Q}$ which are associated with the smallest eigenvalues. The result in (10) can be seen in contrast to the average noise power

$$\sigma_w^2 = \frac{1}{K\ L_g}\ \mathrm{tr}\{\mathbf{R}_{ww}\} \qquad (12)$$

in a subband signal. If the noise is correlated and $\mathbf{G}(z)$ can be constructed to occupy a subspace where the noise is lowest, then the synthesis filter bank decoder can reduce the influence of the channel noise power.

4. FILTER BANK DESIGN

This section aims at designing the synthesis filter bank in a perfectly reconstructing system given the second order statistics of the corrupting noise, in order to minimise the noise power at the decoder output, σ_e^2 as derived in Sec. 3. Hence we can formulate the following

Design Problem. Solve

$$\mathbf{G} = \arg\min_{\mathbf{G}}\{\mathrm{tr}\left\{\mathbf{G}\mathbf{R}_{ww}\mathbf{G}^{\mathrm{H}}\right\}\} \qquad (13)$$

with the coefficient matrix $\mathbf{G}$ as defined in (8), subject to

$$\mathbf{G}(z)\mathbf{H}(z) = z^{-\Delta} \qquad (14)$$

with appropriate $\mathbf{H}(z)$ and delay Δ.

4.1 Constraints

Let us first inspect the constraint in (14). To ensure that $\mathbf{G}(z)$ is invertible, structural constraints can be placed on the polyphase synthesis matrix. As a consequence of a proof in [6] for the analysis filter bank, the existence of an FIR inverse is guaranteed if $\mathbf{G}(z)$ can be factorised as

$$\mathbf{G}(z) = \mathbf{G}_0(z)\prod_{i=1}^{M}\mathbf{V}_i(z) \qquad (15)$$

with $\mathbf{V}_i(z) = \mathbf{I} - \mathbf{u}_i\mathbf{v}_i^{\mathrm{H}} + z^{-1}\mathbf{u}_i\mathbf{v}^{\mathrm{H}}$ and $\mathbf{u}_i,\ \ \mathbf{v}_i \in \mathbb{R}^N$.

A simpler constraint is to force $\mathbf{G}(z)$ to be paraunitary, such that $\mathbf{H}(z) = \mathbf{G}^{\mathrm{H}}(z^{-1})$ is directly available. For $\mathbf{G}(z)$ to be paraunitary, a necessary and sufficient condition is a factorisation

$$\mathbf{G}(z) = \mathbf{G}_0\prod_{i=1}^{M}\mathbf{V}_i(z) \qquad (16)$$

with $\mathbf{V}_i(z) = \mathbf{I} - \mathbf{v}_i\mathbf{v}_i^{\mathrm{H}} + z^{-1}\mathbf{v}_i\mathbf{v}_i^{\mathrm{H}}$ and $\mathbf{v}_i \in \mathbb{C}^K$ with $\|\mathbf{v}_i\| = 1$, which follows as a consequence from a proof on $\mathbf{H}(z)$ in [9].

4.2 General Approach

Limiting the design to a paraunitary filter bank system is a sever restriction and likely to be suboptimal in minimising σ_e^2. However, an unconstrained optimisation scheme can be adopted to search for the components $\mathbf{v}_i$, $i = 1(1)M$, in (16). Note that the order M of the polyphase components may also have to be optimised.

For $M = 0$, the optimal filter bank fulfilling the design problem in (13) and (14) is given by

$$\mathbf{G}_0 = \left[\mathbf{q}_0^{(0)}\ \ \mathbf{q}_1^{(0)}\ \ \ldots\ \ \mathbf{q}_{N-1}^{(0)}\right]^{\mathrm{H}}. \qquad (17)$$

As paraunitarity of $\mathbf{G}(z)$ implies $\mathbf{G}\ \mathbf{G}^{\mathrm{H}} = \mathbf{I}_N$, the rows of $\mathbf{G}$ are unit norm vectors. The trace in (10) can be represented by N Rayleigh quotients of such unit norm rows of $\mathbf{G}$, which are minimised by the eigenvectors corresponding to the N smallest eigenvalues of $\mathbf{R}_{ww}$ [10], as stated in (17).

Note that the cost function (10) is a fourth order polynomial in the elements of the coefficient vectors $\mathbf{v}_i$, $i = 1(1)M$, and hence local minima in (10) are likely to exist with respect to the parameters that need to be optimised. Nevertheless, due to the considerable parameter search space for increasing M, a suboptimal iterative design approach is suggested below.

4.3 Suboptimal Iterative Design Algorithm

The proposed suboptimal design approach is an iterative scheme based on the factorisation in (16), and can be outlined by the following steps:

Initialisation. Set $i = 0$ and determine $\mathbf{R}_{ww}^{(0)} \in \mathbb{C}^{K\times K}$ with modal matrix $\mathbf{Q}^{(0)}$ and eigenvalues in $\mathbf{\Lambda}^{(0)}$ ordered according to (11). The optimal synthesis matrix is then given by the coefficient matrix chosen according to (17).

Iteration. Set $i = i + 1$ and determine $\mathbf{R}_{ww}^{(i)} \in \mathbb{C}^{K(i+1)\times K(i+1)}$ A new polyphase synthesis matrix $\mathbf{G}_i(z)$ is constructed by multiplying a paraunitary factor $\mathbf{V}_i(z)$ to the previous solution $\mathbf{G}_{i-1}(z)$,

$$\mathbf{G}_i(z) = \mathbf{G}_{i-1}(z)\cdot\mathbf{V}_i(z)\quad . \qquad (18)$$

This factor $\mathbf{V}_i(z)$ is chosen such that

$$\mathbf{v}_i = \arg\min_{\mathbf{v}_i}\{\mathrm{tr}\left\{\mathbf{G}_i\mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{\mathrm{H}}\mathbf{G}_i^{\mathrm{H}}\right\}\} \qquad (19)$$

where $\mathbf{G}_i$ is the coefficient matrix of $\mathbf{G}_i(z)$ analogous to the definition in (8).

The iteration can be stopped after M iterations once a specified noise variance σ_e^2 has been reached, or if no further reduction of σ_e^2 by increasing the filter order appears possible. Clearly the above algorithm is by no means optimal; in particular having a globally optimal solution at step i does not imply that despite global optimisation of $\mathbf{v}_{i+1}$ the resulting $\mathbf{G}_{i+1}(z)$ is also globally optimal. However, the results obtained from it can still motivate the merits of the general approach.

In our implementation, the optimisation at the ith step of the iterative design scheme in (19) is accomplished by an iterative simplex search algorithm. The simplex search is started with the result of a random search for an initial vector at step i, to avoid local minima. In the following, we present some selected design results.

5. DESIGN EXAMPLE

In our simulation scenario, we consider a $N/K = 1/3$ channel codec with $N = 2$ and a transmission over $K = 6$ separate channels. The channel noise is mutually uncorrelated with respect to the channels, i.e. the covariance matrices $\mathbf{R}_{ww}^{(i)}$ are block diagonal, however noise on the first four channels has a smaller variance and is spectrally shaped, as evident from the power spectral densities in Fig. 5.

As evident from the adapted magnitude responses, the filter bank design algorithm deselects the two channels with the highest noise power from transmission, which is already

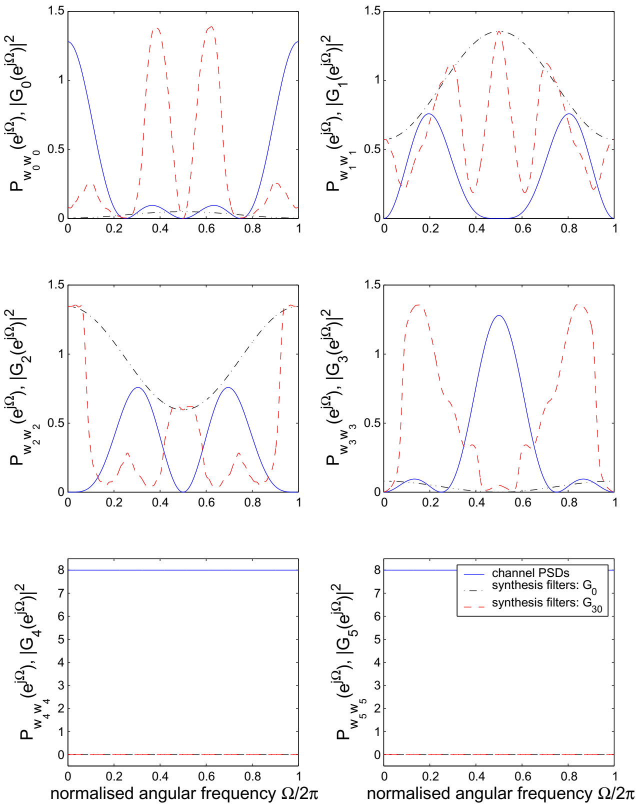

Figure 5: Noise power spectral densities and magnitude responses of the designed synthesis filters at design stages $m = 0$ and $m = 30$, for all $K = 6$ transmission channels.

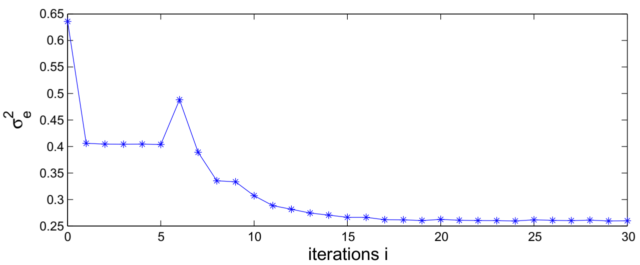

Figure 6: Evolution of the cost function value σ_e^2.

accomplished in the initialisation step $m = 0$. Thereafter, as the cost function evolution in Fig. 6 shows, the correlation within each channel if further exploited to reduce the received noise power σ_e^2, and the transmission balanced over the first four channels. It is also interesting to note that the cost function is not monotone decreasing.

6. DISCUSSION

The oversampling ratio K/N describes the amount of redundancy introduced into the system. If there is any correlation between the K channels, or the noise power on different channels varies, then the K eigenvalues of the covariance matrix $\mathbf{R}_{ww}^{(0)} \in \mathbb{C}^{K\times K}$ are unbalanced. The OSFB codec can exploit this by utilising the eigenvectors corresponding to the N smallest eigenvalues to construct the synthesis filter bank according to (17).

Increasing the filter length L_p does not affect the redundancy of the codec, but can lead to further reduction of the noise variance in the receiver, since the eigen spectrum of the covariance matrix $\mathbf{R}_{ww}^{(i)}$, $i \geq 1$ is finer resolved. The OSFB codec can take this into account by additionally shaping the transmitted signal in order to transmit in a low-noise subspace, as demonstrated in Sec. 5. However, increasing L_p potentially also introduces more noise into the encoder, similar to trade-off considerations for the length of a linear equaliser [11].

7. CONCLUSIONS

We have considered the use of oversampled filter banks in the context of channel coding, and presented a constrained optimisation scheme to minimise, subject to the filter banks being perfectly reconstructing, the noise power at the decoder output. We have proposed to implicitly fulfil the constraints by optimising factorisations of the polyphase synthesis matrix that guarantee its paraunitarity. A suboptimal iterative scheme was proposed and demonstrated. Both the example presented here and results in [3, 6] suggest that OSFBs can be powerful tools for channel coding. Therefore, our current work focuses on alternative or re-iterated update schemes, in order to obtain a globally optimal filter bank design that can fully exploit any correlations in the channel noise.

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