# THE INFLUENCE OF THE NON-UNIFORM SPLINE BASIS ON THE APPROXIMATION SIGNAL

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## ABSTRACT

This paper is concerned with the problem of recovering a discrete signal from a set of irregularly spaced samples. The approximation method is based on spline functions using non-uniform B-splines. According to the various knot sequence configurations, several bases can be used. We study the important issue of selecting an adequate basis of the spline function. The analysis shows that the choice of the elements and dimension of the basis has a strong influence on the quality of the signal approximation. For a given degree of the spline and for a particular knot sequence configuration, the smallest dimension of the basis provides good performances compared to the basis spline of higher dimension. Moreover this basis, with the smallest dimension, requires only two consecutive knots for the construction of its elements. The theoretical results are illustrated with examples.

## 1. INTRODUCTION

The majority of the theoretical tools developed in the field of digital signal processing are based on a uniform distribution of the samples. However, the non-uniform sampling problem arises naturally in many scientific fields such as geophysics, astronomy, meteorology, medical imaging, computer vision …
In this paper, we are interested in the problem of recovering a discrete signal from its irregularly spaced samples. The proposed study is carried out within the framework of future investigation in the topic of data compression. The redundancy of information can be controlled by a variable sampling rate. Indeed, the sampling rate of a signal is adapted to its instantaneous frequencies involving a non-uniform distribution of samples. We suppose that the sample locations are known. The interval of time, between two consecutive samples, is supposed to be an integer multiple of some underlying sampling period of the discrete signal. Among the significant number of reconstruction methods, we retain the interpolation methods related to piecewise polynomial functions. Moreover, we focus on the polynomial functions based on non-uniform B-spline functions because they provide many interesting properties [1]. We study how the elements and the dimension of the spline basis influence the quality of the approximation.
This paper is organized as follows. The next section, section 2, introduces elementary properties of the non-uniform B-spline functions. Section 3 presents the spline function based on non-uniform B-splines. The approximation method is developed in section 4. According to selected knot sequence configurations, several spline bases can be built. The evaluation of the B-spline coefficients is explicitly given. The influence of the dimension and the elements of the spline basis are investigated. The upper bounds of the estimate error are provided. A difficulty arises when some parameters have to be estimated: in fact, some of the interpolation methods require the knowledge of derivatives of the signal. We show that a simple procedure allows accurate estimate. Section 5 illustrates the theoretical results and the approximation procedures through some examples. Section 6 concludes.

## 2. BASIC PROPERTIES OF THE NON-UNIFORM B-SPLINE FUNCTIONS

In this section, we review the basic properties of the non-uniform B-spline functions. The definition of a non-uniform B-spline function has been proposed initially by Curry and Schoenberg [1]. Given a set of $k+2$ samples, located at known knots. The knots sequence $t$ is organized according to an increasing order $t_j < \ldots < t_{j+k+1}$. For $x \in R$, the $jth$ non-uniform B-spline function of order $k+1$ (degree $k$), is denoted either $B_{j,k,t}(x)$ or $B_{j,k}(x)$. It is given by the following equation:

$$B_{j,k,t}(x) = (t_{j+k+1} - t_j)[t_j,\ldots,t_{j+k+1}](.-x)_+^k \qquad (1)$$

This equation is based on the $(k+1)th$ divided difference applied to the function $(.-x)_+^k$. The definition of the divided difference is as follows:

$$[t_j,\ldots,t_{j+k+1}](.-x)_+^k =$$
$$(t_{j+k+1} - t_j)^{-1}\left([t_{j+1},\ldots,t_{j+k+1}](.-x)_+^k - [t_j,\ldots,t_{j+k}](.-x)_+^k\right)$$

where $(t-x)_+ = \max(t-x,0)$ is the truncation function.
The B-spline function is represented by a piecewise polynomial of degree $k$. It is a positive function on the interval $]t_j,\ldots,t_{j+k+1}[$ and has a finite support. If a knot in the sequence $t$ has a multiplicity of order $\mu+1$, i.e. the knot occurs $\mu+1$ times, then the definition of the divided difference applied to the function $g=(.-x)_+^k$ becomes:

$$[t_0,\ldots,t_\mu]g = \frac{g^{(\mu)}(t_0)}{\mu!} \quad \text{if} \quad t_0 = \ldots = t_\mu \qquad (2)$$

Equation (3) relates the multiplicity $\mu_j$ of a knot $t_j$ to the number of times $(r_j - 1)$ that the B-spline function is continuously differentiable at the knot $t_j$:

$$r_j + \mu_j = k+1 \qquad (3)$$

## 3. THE NON-UNIFORM SPLINE FUNCTIONS

The linear combination of the non-uniform B-spline functions $\{B_{j,k,t}(x)\}$ of degree $k$, associated to the knot sequence $t$, is called the spline function. It is given by:

$$f(x) = \sum_j a_j B_{j,k}(x) \quad (4)$$

where $\{a_j\}$ are the B-spline coefficients, and $x$ is the knot for which the sample value will be evaluated.

Let $t$ be a strictly increasing sequence $\{t_1,...,t_{l+1}\}$, and a nonnegative integer sequence $r = \{r_2,...,r_l\}$ with $r_j \le k$ for $j = 2,...,l$. The spline space is denoted $\prod_{k,t,r}$. It is the linear space spanned by the polynomials of degree $k$ associated to the knot sequence $t$ and satisfying the $r$ relative conditions to the derivative continuity. The dimension of the spline space is given by the following equation [1, 6]:

$$n = (k+1)l - \sum_{i=2}^{l} r_i \quad (5)$$

The set of the non-uniform B-splines $\{B_{j,k},...,B_{j+n-1,k}\}$ of degree $k$ generates a basis for the space $\prod_{k,t,r}$. For a given degree $k$, equation (5) means that the elements and the dimension of the spline basis are closely related to (i) the length of the selected knot sequence and (ii) the multiplicity of each knot in the sequence. Thus, the representation of a spline function in terms of B-splines is not unique. Before developing the approximation methods for a signal reconstruction, we describe in the following sub-sections, the different spline functions obtained on three particular knot sequence configurations denoted *Seq* 1, *Seq* 2 and *Seq* 3.

**Case 1: The first knot sequence configuration**

We start with the traditional spline function. Each knot belonging to the interval $[a_1, b_1]$ has a multiplicity of order 1. The knot sequence, denoted *Seq* 1, is as follows:

$$a_1 = t_1 < ... < t_{k+1} < ... < t_{n+1} < ... < t_{n+k+1} = b_1 \qquad Seq\ 1$$

The spline space is spanned by the combination of $k+1$ B-spline functions $\{B_{1,k},...,B_{n,k}\}$ on the intersection interval $[t_{k+1}, t_{k+2}]$. These B-splines have been extensively studied for a uniform repartition of knots [2, 3, 4].

**Case 2: The second knot sequence configuration**

A multiplicity of order $k+1$ is imposed to the first and the last knots of the sequence defined on the interval $[a_2, b_2]$. While each knot inside the interval has a multiplicity of order one. The knot sequence, denoted *Seq* 2, is given as follows:

$$a_2 = t_1 = ... = t_{k+1} < ... < t_{n+1} = ... = t_{n+k+1} = b_2 \qquad Seq\ 2$$

The spline space is spanned by the combination of $n$ B-spline functions $\{B_{1,k},...,B_{n,k}\}$. According to equation (5), the spline basis dimension depends on the number of knots available inside the interval $]a_2, b_2[$.

**Case 3: The third knot sequence configuration**

The sequence is composed only of two consecutive knots. Each knot has a multiplicity of order $k+1$. The knot sequence, denoted *Seq* 3, is presented as follows:

$$a_3 = t_1 = ... = t_{k+1} < t_{n+1} = ... = t_{n+k+1} = b_3 \qquad Seq\ 3$$

For a given degree $k$, this configuration of knots provides the smallest spline space dimension equal to $k+1$. Whatever the degree of the spline, the construction of the basis elements has been generalized [1]. The $k+1$ first knots are renamed by $\lambda_j$ ($\lambda_j = t_j = t_{j+1} = ... = t_{j+k}$) and the $k+1$ last knots by $\lambda_{j+1}$ ($\lambda_{j+1} = t_{j+k+1} = ... = t_{j+k+n}$). For any degree $k$ of the spline, the B-spline functions are generalized by the following equation:

$$B_{i,k}(x) = C_k^i \left(1 - \frac{x - \lambda_j}{\lambda_{j+1} - \lambda_j}\right)^{k-i} \left(\frac{x - \lambda_j}{\lambda_{j+1} - \lambda_j}\right)^i$$

for $0 \le i \le k$ and $\lambda_j \le x \le \lambda_{j+1}$.

where $C_k^i = k!/(i!(k-i)!)$ is the binomial coefficient. For the special values of the knots, $\lambda_j = 0$ and $\lambda_{j+1} = 1$, the basis elements are identical to the Bernstein polynomials of any degree $k$:

$$B_i^k(x) = C_k^i (1-x)^{k-i} x^i \ (0 \le i \le k) \text{ and } 0 \le x \le 1$$

In this special case, the spline function is called Bézier curve and is used extensively in the CAGD [6].

## 4. THE APPROXIMATION METHOD

The reconstruction method is based on an interpolation method using spline functions. The method is closely related to the process of evaluating the B-spline coefficients so that the resulting spline function satisfies the imposed criteria. In the literature, many approximation methods have been developed. The most popular one is used in the CAGD. It is known as the variation diminishing spline approximation [5]. The B-spline coefficients are given directly by the signal values defined at the control points (knot averages of the sequence $t$) of the spline. In this paper we try to keep the number of information taken from the signal as small as possible, hence we do not use this method. Several questions arise concerning the quality of the reconstructed signal. In the following sub-sections, we study and compare the performance of the approximation method based on each knot sequence configuration (*Seq* 1, *Seq* 2, *Seq* 3).

### 4.1 Evaluating the B-spline coefficients

Given a set of sampled discrete signal values $\{y(t_i)\}$ defined at non-uniform knots $\{t_i\}$, the problem consists in finding a spline function $f$ in the spline space. This results in the evaluation of the $n$ unknown B-spline coefficients $\{a_j\}$ such that the spline function satisfies the interpolation conditions:

$$f(t_i) = \sum_{j=m}^{n-m+1} a_j B_{j,k}(t_i) = y(t_i) \quad \text{for} \quad i = m,...,n-k+m$$

where:

$m = k+1$, if the selected knot sequence corresponds to the description given by the case 1 (*Seq* 1);

$m = 1$, if the selected knot sequence corresponds to the description given by the cases 2 and 3 (*Seq* 2, *Seq* 3).

These interpolation conditions provide $n-k+1$ equations. Thus, $k-1$ other equations are necessary. We supplement the number of equations to $n$ with the successive *qth* order derivatives of the spline function evaluated at the first and last knots where the spline basis is defined:

$$f^{(q)}(t_i) = \sum_{j=m}^{n-m+1} a_j B_{j,k}^{(q)}(t_i) = y_i^{(q)}(t_i) \quad \text{for} \quad i = m \text{ and } i = n-k+1.$$

where $q$ is the derivative order of the spline function. It is determined by $q = \lfloor (k-1)/2 \rfloor$ (where $\lfloor . \rfloor$ is the floor function). Therefore, the $n$ B-spline coefficients are evaluated by solving the linear system of $n$ equations.

### 4.2 Approximation errors

In this section, we analyze and compare the performance of the approximation methods. We introduce briefly some classical results such as the distance of a function to a polynomial. To measure the approximation error, we use the uniform norm which is a bounded function $g$ defined on an interval $[a,b]$:

$$\|g\|_{[a,b]} = \sup_{a \le x \le b} |g(x)|$$

Let us consider a given function $g$ in $C^{k+1}[a,b]$ (i.e. $g$ has $k+1$ continuous derivatives on $[a,b]$). The approximation of the function by a polynomial of degree $k$ involves an error which is referred to as the distance from $g$ to the space $\pi_k$ of polynomials of degree $\le k$ and is defined as follows [5]:

$$dist_{[a,b]}(g,\pi_k) = \inf_{p \in \pi_k} \|g\text{-}p\|_{[a,b]}$$

The upper bound for the distance from $g$ to polynomials of degree $k$ is:

$$dist_{[a,b]}(g,\pi_k) = \|g-p\|_{[a,b]} \le K_k h^{k+1} \|d^{k+1} g\|_{[a,b]}$$

where $h = b - a$ and the constant $K_k = \left[2^{k+1}(k+1)!\right]^{-1}$ depends only on the degree $k$ of the polynomial [5].

We focus, in this section, on the upper bound of the distance from the signal $g$ to the spline space ($\|g-f\|$). Therefore we need to compute the upper bound of the spline function. It is closely related to the B-spline coefficients. According to the properties of the B-splines we get:

$$|f(x)| \le \left| \sum_{j=m}^{n-m+1} a_j B_{j,k}(x) \right| \le \sum_{j=m}^{n-m+1} |a_j| \qquad (6)$$

The resolution of the linear system, described in sub-section 4.1, shows that the B-spline coefficients are given by a weighted combination of the signal values and its derivative values. The weights depend on the distance between consecutive knots belonging to the same knot sequence. Therefore, it is important to have a good approximation of the derivatives.

It is well known that when the degree of the spline function increases the approximation error decreases [1, 5]. For this reason, we are interested more particularly on the elements and the dimension of the spline basis for a given degree $k$. Indeed, for a fixed degree, the spline function can be represented by many bases of equal or different dimensions. The goal of this section is to study the influence of these bases on the quality of the reconstructed signal. We propose to follow two steps.

**First step:** The degree of the spline function is fixed, while its basis dimension varies. Increasing the dimension of the spline basis, results in widening the definition field $[a_2, b_2]$ of the spline. This change requires the evaluation of the new B-spline coefficients. We note that these coefficients depend not only on the $n-k+1$ information related to the signal values, but also on the derivatives of the signal. The upper bound of the spline function, given by equation (6), is then affected and involves an increase of the upper bound of the approximation error. We note the approximation error $\|y-f\|_{[a,b]}$ as a function $\varepsilon(k,n,Seq\,\mathrm{i})$, depending on the spline degree $k$, its dimension $n$ and the knot sequence configuration. We set $h_i = t_{j+i} - t_{j+i-1}$ for $i = 1,\ldots,n+k+1$ and $h = b_i - a_i$ for $i = 1,2,3$ according to the selected sequence.

Some upper bounds of the approximation errors are given for $k = 2$ and some dimensions $n = 3,4,5,6$.

- $\varepsilon(2,3,Seq\,3) \le \dfrac{7h_1^3}{48}\|d^3 y\|_{[a,b]} = \dfrac{7h^3}{48}\|d^3 y\|_{[a,b]}$
- $\varepsilon(2,4 Seq\,2) \le \left(\left(4 + \dfrac{2h_2}{h_1}\right)\dfrac{h^3}{48} + (h_1 + h_2)\dfrac{h^2}{16}\right)\|d^3 y\|_{[a,b]}$
- $\varepsilon(2,5,Seq\,2) \le \left(\left(4 + \dfrac{2(h_2+h_3)}{h_1} + \dfrac{2h_3}{h_2}\right)\dfrac{h^3}{48} + (h_1 + h_2 + h_3)\dfrac{h^2}{16}\right)\|d^3 y\|_{[a,b]}$
- $\varepsilon(2,6,Seq\,2)$

$$\le \left(\left(4 + \frac{2(h_2+h_3+h_4)}{h_1} + \frac{2(h_3+h_4)}{h_2} + 2\frac{h_4}{h_3}\right)\frac{h^3}{48} + (h_2 + h_3 + h_4)\frac{h^2}{16}\right)\|d^3 y\|_{[a,b]}$$

The comparisons of these upper bounds show that the smallest one is given for the smallest dimension of the spline space ($n = 3$). It corresponds to the third knot sequence configuration *Seq* 3. These results remain valid for degrees higher than 2.

**Second step:** The degree of the spline function and the dimension of the basis are fixed. Moreover, the dimension is equal to the smallest possible value, i.e. $k+1$ (corresponding to the first and third knot sequence configuration (*Seq* 1 and *Seq* 3). In the third knot sequence configuration (*Seq* 3), the first and the last B-spline coefficients of the spline are respectively equals to $y(t_1)$ and $y(t_{n-k+1})$ whatever the degree of the spline. While, for the knot sequence configuration *Seq* 1, these coefficients depend not only on $y(t_{k+1})$ and $y(t_{k+2})$ but also on the derivative values of the signal. This difference affects the upper bound of the approximation error. Some upper bounds of the approximation errors are given for $k = 2$ and $n = 3$.

- $\varepsilon(2,3,Seq\,3) \le \dfrac{7h_1^3}{48}\|d^3 y\|_{[a,b]} = \dfrac{7h^3}{48}\|d^3 y\|_{[a,b]}$
- $\varepsilon(2,3,Seq\,1) \le \left(\left(4 + \dfrac{2h_4}{h_3}\right)\dfrac{h^3}{48} + (h_2 + h_3 + h_4)\dfrac{h^2}{16}\right)\|d^3 y\|_{[a,b]}$

We deduce that the second upper bound of the approximation error is greater than the first one. These remarks remain valid for higher degrees.

## 5. EXPERIMENTAL RESULTS

In this section, we present some experimental results to illustrate how the spline basis influences the performance of the signal approximation.

Let us remember that the resolution of the linear system requires the $qth$ order successive derivative values at the first and last knots of the sequence where the spline function is defined. The signal ($y(t)$) to be interpolated is known only through some data points $\{y(t_i)\}$. The $qth$ order derivative values can be approximated either by this traditional equation $y^{(q)}(t_i) = d^{(q)}y(t_i) / d^{(q)}t_i = d^{(q-1)}(y(t_{i+1}) - y(t_i)) / d^{(q-1)}(t_{i+1} - t_i)$ or with a polynomial of degree $k$ on the interval $[t_m, t_{n-k+1}]$ where the spline function basis is defined. In this case, we chose to estimate the $qth$ order derivative values, in any knots belonging to the interval, from this polynomial.

We generate a discrete signal which is then randomly sub-sampled. For different percentage of samples to be interpolated, we reconstruct this signal from a spline function of degree 5 using the knot sequence configurations *Seq* 1 and *Seq* 3. Remember that in these cases, the interpolation is carried out by two different spline bases having the same dimension. Figure 1 provides the maximal reconstruction error (in dB) for a given non-uniform distribution of samples. The figure shows that better approximation results are obtained when we approximate the derivative values from a polynomial of degree 5. Moreover, to check the performance of the derivative approximations we have replaced those by the theoretical derivative values of the signal. Figure 1 shows that the theoretical ones introduce differences between the interpolation methods.

Figure 2 provides the maximal approximation errors versus sampling steps and different dimensions $n = 4, ..., 9$. These simulations were carried out with the estimate derivatives given using a polynomial of degree 3. We see that the dimension of the spline basis influences on the interpolation signal. For a given spline function of degree 3, we note that the basis of the smallest dimension (*Seq* 3) gives better results than higher dimensions (*Seq* 2). Table 1 compares the information necessary to interpolate a signal from each of the 3 sequences, on the same interval $[t_{k+1}, t_{k+2}]$. Let us take for example the function spline of degree 5. The sequence 1 uses 12 knots to build the elements of its basis, 2 samples, 2 first derivative values and 2 second derivative values. While the sequence 3 use only 2 knots for the construction of its base, 2 samples, 2 first derivative values and 2 second derivative values on the same interval. The reconstruction method based on the sequence 1 requires $2k$ additional knots compared to sequence 3. Note that in our case, all this information were estimated from the same total number of samples of the signal.

| Sequences | Knots | Samples | Derivatives |
|---|---|---|---|
| *Seq* 1 | $2(k+1)$ | 2 | $q = \lfloor (k-1)/2 \rfloor$ |
| *Seq* 2 | $n-k+1$ | $n-k+1$ | $q = \lfloor (k-1)/2 \rfloor$ |
| *Seq* 3 | 2 | 2 | $q = \lfloor (k-1)/2 \rfloor$ |

Table 1: Comparisons of the required information according to the sequences

## 6. CONCLUSIONS AND PERSPECTIVES

This paper is concerned with the problem of recovering a discrete signal from a set of irregularly spaced samples. The reconstruction method is an interpolation method based on non-uniform B-spline functions. We showed that the knot sequence configuration influences on the quality of the reconstructed signal. Among the three knot sequence configurations, in absence of noise, we privilege the sequence which is built only from two consecutive knots (*Seq* 3). In this sequence each knot has a multiplicity of order $k+1$. The corresponding basis is easily constructed. The elements of this basis depend only on these two consecutive knots, whatever the degree of the spline. While for the two other sequences (*Seq* 1 and *Seq* 2), as soon as the degree of the spline increases, the number of knots to be used for the construction of the B-spline elements is significant. Moreover their computation becomes difficult. With a good approximation of the derivative values, the interpolation method provides interesting results in term of the quality of the approximated signal. Unfortunately, all these approximation methods, whatever the basis spline, are sensitive to the presence of noise. This is due to the derivatives required by the B-spline coefficients. Indeed, it is well known that the derivative amplifies the presence of noise in a signal. It is thus necessary in this situation to find the B-spline coefficients according to an approximation method which minimizes, for example the reconstruction least mean square error. This will be the topic for further work.

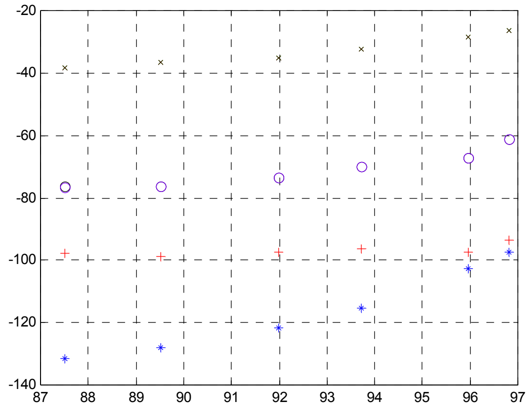


Figure 1: The maximal interpolation errors (in dB) versus the percent of samples to be interpolated; with: theoretical derivatives for (*) *Seq* 3 and (+) *Seq* 1; traditional approximation derivatives for (o) *Seq* 3 and *Seq* 1; polynomial approximation derivatives for (x) *Seq* 3 and *Seq* 1.

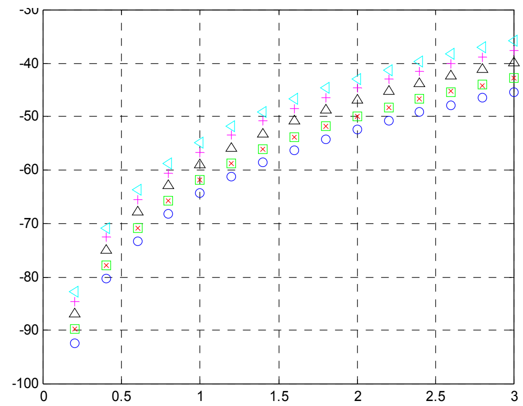


Figure 2: The maximal interpolation error (in dB) versus sampling step: for the dimensions $n = 4$ (o), $n = 5$ (□), $n = 6$ (x), $n = 7$ ( Δ ), $n = 8$ (+) and $n = 9$ (◁).

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