FILTER BANK DESIGN WITH GROUP DELAY APPROXIMATIONS

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ABSTRACT

Multi-rate adaptive filters have been used in various applications with numerous advantages such as low computational load, fast convergence and build in parallelism allowing efficient hardware implementation. Drawbacks when using multi-rate processing are mainly related to aliasing and reconstruction effects. In this paper, a filter bank design method using multi-criteria including inband aliasing, residual aliasing, magnitude and phase constraints on the total filter bank is proposed. The analysis filter bank is first designed with minimum inband aliasing and approximately linear phase in the passband. From a given analysis filter bank, the synthesis filter bank is designed with minimum residual aliasing between subbands while controlling the amplitude and delay distortion level for each frequency component directly. Accurate approximations for the group delay errors are derived for both designed problems. By employing these approximations, the multi-criteria optimization problem can be efficiently formulated as quadratic optimization problems. A design example shows that the group delay approximations are highly accurate with the group delay errors restricted to small values.

1. INTRODUCTION

Multi-rate signal processing is gaining more and more importance in a wide range of applications such as echo cancellation, audio coding, video coding, signal compression, microphone array, speech enhancement and equalisation [1], [2]. In multi-rate processing, the signal to be processed is divided into subbands by using an analysis filter bank and then decimated according to the new bandwidth of the subbands. The decimation in combination with non-perfect filters in the filter bank results in aliasing of the subband signals. It is possible to cancel aliasing perfectly in the synthesis filter bank when the whole multi-rate chain is designed to yield no distortion, i.e. the total transfer function is reduced to a simple delay. This is often referred to as the perfect reconstruction property [3]. However, any filtering operation in the subbands will cause phase and amplitude changes, thereby altering this property. Thus, there is a need to optimize both the inband aliasing and the residual aliasing effects while controlling the distortion levels in the filter bank. Furthermore, the delay for the filter bank has a critical impact in many applications such as telephony. Consequently, it is important to include a delay specification in the optimization problem.

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In [4], [5] an approach for the design of nearly perfect reconstruction filter bank is presented. In [6], the aliasing effect and the distortion are combined into a quadratic optimization cost and a simple quadratic programming is used to optimize the analysis and the synthesis prototype filters. The problem of controlling the phase and group delay for the total filter bank is investigated in [7] and an adaptive non-linear optimization method is used to solve the problem. That formulation, however, does not include a capability to directly control the group delay. Thus, in this paper we propose and derive efficient and highly accurate approximations for the group delay errors to allow exact control of the phase and group delay. These approximations are shown to be linear functions of the prototype filters. Consequently, additional linear constraints are incorporated into the optimization problem to allow approximately linear phase for the total filter bank. The analysis and synthesis filter banks designed using multi-criteria including inband aliasing, residual aliasing, magnitude and phase constraints on the total filter bank is then proposed. These criteria can be controlled exactly depending on the practical application. The analysis filter bank is initially designed with minimum inband aliasing in conjunction with frequency domain specification and approximately linear phase in the passband. From a given analysis filter bank, the synthesis filter bank is designed with minimum residual aliasing while constraining the distortion level and the group delay for all the frequencies. Consequently, the filter bank designed problems are formulated as semi-infinite quadratic optimization problems. A design example shows that the group delay approximations are highly accurate with the group delay errors restricted to small values.

2. FILTER BANK STRUCTURE

Uniformly modulated filter bank (UMF) is formed by modulated versions of the analysis and synthesis prototype filters. Denote $\mathbf{h} = [h(0), \cdots, h(L_a - 1)]^T$ as impulse response of the analysis prototype filter of length L_a with the corresponding transfer function

$$H(z) = \mathbf{h}^T \phi_a(z) \tag{1}$$

where $\phi_a(z) = [1, z^{-1}, \cdots, z^{-(L_a-1)}]^T$. For an UMF system with M subbands, the subband analysis filters $H_m(z)$, $0 \leq m \leq M-1$, are obtained from the prototype filter $H(z)$ as follows:

$$H_m(z) = H(zW_M^m) \tag{2}$$

where $W_M = e^{-j2\pi/M}$. The input signal $X(z)$ is filtered by the m^{th} analysis filter $H_m(z)$ and decimated by a factor D,

$D \leq M$, according to

$$X_m(z) = \frac{1}{D}\sum_{d=0}^{D-1} H(z^{1/D}W_M^m W_D^d)X(z^{1/D}W_D^d) \tag{3}$$

where $W_D = e^{-j2\pi/D}$. The output from the analysis filter bank is passed through interpolators with compressing effect

$$Y_m(z) = X_m(z^D). \tag{4}$$

These signals are filtered by the synthesis filter bank and then added to form the output signal $Y(z)$. Denote $\mathbf{g} = [g(0), \cdots, g(L_s-1)]^T$ as the impulse response of the synthesis prototype filter of length L_s with the corresponding transfer function

$$G(z) = \mathbf{g}^T \phi_s(z) \tag{5}$$

where $\phi_s(z) = [1, z^{-1}, \cdots, z^{-(L_s-1)}]^T$. The subband synthesis filter is obtained from the prototype filter as $G(z)$ as follows

$$G_m(z) = G(zW_M^m). \tag{6}$$

The output signal can be expressed as

$$\begin{aligned} Y(z) &= \tfrac{1}{D}\sum_{d=0}^{D-1} X(zW_D^d) \sum_{m=0}^{M-1} H(zW_M^m W_D^d)G(zW_M^m) \\ &= \tfrac{1}{D}X(z)\sum_{m=0}^{M-1} H(zW_M^m)G(zW_M^m) \\ &+ \tfrac{1}{D}\sum_{d=1}^{D-1} X(zW_D^d)\sum_{m=0}^{M-1} H(zW_M^m W_D^d)G(zW_M^m). \end{aligned} \tag{7}$$

A direct form realization of an analysis and synthesis filter bank is given in Fig. 1.

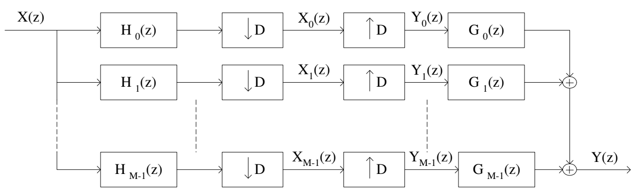

Figure 1: Direct form realization for an UMF.

3. ANALYSIS FILTER BANK DESIGN

The analysis prototype filter is designed with minimum inband aliasing subject to frequency response constraints in the passband. Due to the structure of UMF, it is only necessary to optimize the inband aliasing for the 0^{th} subband [7]. The total inband aliasing energy for all the frequencies $\omega \in [-\pi, \pi]$ when assuming a white input signal can be given as

$$\beta(\mathbf{h}) = \frac{1}{2\pi D^2}\int_{-\pi}^{\pi}\sum_{d=1}^{D-1} |H(e^{j\omega/D}W_D^d)|^2 d\omega. \tag{8}$$

This inband aliasing can be expressed as a quadratic function of the analysis prototype filter $\mathbf{h}$ as follows

$$\beta(\mathbf{h}) = \mathbf{h}^T\mathbf{B}\mathbf{h} \tag{9}$$

where

$$\mathbf{B} = \frac{1}{2\pi D^2}\sum_{d=1}^{D-1}\int_{-\pi}^{\pi} \phi_a(e^{j\omega/D}W_D^d)\phi_a^H(e^{j\omega/D}W_D^d)d\omega \tag{10}$$

and $(\cdot)^H$ denotes the Hermitian operation of a vector. In the sequel, we will formulate the frequency response constraints for the analysis prototype filter in the passband [8]. Denote the desired frequency response of the analysis prototype filter in the passband as

$$\hat{H}(e^{j\omega}) = e^{j\hat{\theta}_H(\omega)},\ \forall \omega \in [-\omega_p, \omega_p] \tag{11}$$

where $\hat{\theta}_H(\omega) = -\omega\hat{\tau}_H$ with $\hat{\tau}_H$ is the desired constant group delay and $[-\omega_p, \omega_p]$ is the passband.

The analysis filter bank is constrained to the desired response in the passband according to

$$|H(e^{j\omega}) - \hat{H}(e^{j\omega})| \leq \varepsilon_H,\ \forall\ \omega \in [-\omega_p, \omega_p] \tag{12}$$

where ε_H is a small specified error. Denote $\phi_a^r(\omega)$ and $\phi_a^i(\omega)$ as the real and imaginary parts of $\phi_a(\omega)$, respectively. Assuming that (12) is achieved then for a small ε_H we may use the following approximations

$$\mathbf{h}^T\phi_a^r(\omega) \approx \cos(\omega\hat{\tau}_H),\ \ \mathbf{h}^T\phi_a^i(\omega) \approx -\sin(\omega\hat{\tau}_H), \tag{13}$$

and

$$A_H(\omega) \approx 1,\ \ \forall\ \omega \in [-\omega_p, \omega_p], \tag{14}$$

where $A_H(\omega)$ is the magnitude response of the analysis filter bank. Let $\theta_H(\omega)$ and $\tau_H(\omega)$ denote, respectively, the phase and the group delay of the analysis filter bank. A design objective is to ensure that

$$|\hat{\tau}_H - \tau_H(\omega)| \leq \varepsilon_{H,\tau},\ \forall\ \omega \in [-\omega_p, \omega_p], \tag{15}$$

where $\varepsilon_{H,\tau}$ is small. The group delay error for the analysis filter bank each frequency $\omega \in [-\omega_p, \omega_p]$ can be given as

$$\begin{aligned} e_H(\omega) &= \hat{\tau}_H - \tau_H(\omega) \\ &= \hat{\tau}_H + \tfrac{d\theta_H(\omega)}{d\omega}, \end{aligned} \tag{16}$$

where

$$\theta_H(\omega) = \arcsin\left(\frac{\mathbf{h}^T\phi_a^i(\omega)}{\sqrt{(\mathbf{h}^T\phi_a^r(\omega))^2 + (\mathbf{h}^T\phi_a^i(\omega))^2}}\right). \tag{17}$$

Using approximations (13) and (14), it can be shown that the group delay error can be approximated to a linear function of the analysis prototype filter as

$$e_H(\omega) \approx \mathbf{h}^T\Gamma(\omega), \tag{18}$$

where $\Gamma(\omega)$ is an $L_a \times 1$ vector with the n^{th} element, $0 \leq n \leq L_a - 1$, given as:

$$\Gamma_n(\omega) = I\{je^{-j\omega n - j\hat{\theta}_H(\omega)}(-n + \hat{\tau}_H)\} \tag{19}$$

and $I\{\cdot\}$ is the imaginary part of a complex number. The optimization problem can then be formulated as

$$\begin{cases} \min\limits_{\mathbf{h}} \mathbf{h}^T\mathbf{B}\mathbf{h} \\ |H(e^{j\omega}) - \hat{H}(e^{j\omega})| \leq \varepsilon_H,\ \forall\omega \in [-\omega_p, \omega_p] \\ |\mathbf{h}^T\Gamma(\omega)| \leq \varepsilon_{H,\tau},\ \forall\omega \in [-\omega_p, \omega_p]. \end{cases} \tag{20}$$

Using the real rotation theorem [9], (20) can be reformulated as a linear quadratic programming problem as follows:

$$\begin{cases} \min_{\mathbf{h}} \mathbf{h}^T \mathbf{B} \mathbf{h} \\ R\left\{(H(e^{j\omega}) - \hat{H}(e^{j\omega}))e^{j2\pi\lambda}\right\} \le \varepsilon_H, \\ \quad \forall\, \omega \in [-\omega_p, \omega_p],\ \lambda \in [0,1] \\ |\mathbf{h}^T \Gamma(\omega)| \le \varepsilon_{H,\tau},\ \forall \omega \in [-\omega_p, \omega_p], \end{cases} \tag{21}$$

where $R\{\cdot\}$ denotes the real part. The problem (21) is a semi-infinite quadratic optimization problem with respect to the analysis prototype filter $\mathbf{h}$. This problem can be solved using semi–infinite quadratic programming or the conventional quadratic programming using discretization. In this paper, the second method is chosen for solving (21) for its simplicity. The problem is discretized by restricting ω and λ to finite sets Ω and Λ, respectively. If the unit circle, generated by $e^{j2\pi\lambda}$, $\lambda \in [0,1]$ is approximated by an octagon, the complex error magnitude is, at worst, 0.68 dB deviation from the constraints [9]. With the octagon approximation, (21) becomes a finite quadratic programming problem, which can be solved efficiently by using the standard simplex algorithm.

4. SYNTHESIS FILTER BANK DESIGN

Given an analysis prototype filter designed in Section 3, the synthesis prototype filter is optimized with minimum residual aliasing in conjunction with constraints on the total distortion of the filter bank. Since the first term in (7) can be viewed as the transfer function that filters the original input signal, the total transfer function for the filter bank is given as

$$T(z) = \frac{1}{D}\sum_{m=0}^{M-1} H(zW_M^m)G(zW_M^m) = \mathbf{h}^T \Psi(z)\mathbf{g}, \tag{22}$$

where $\Psi(z) = \frac{1}{D}\sum_{m=0}^{M-1} \phi_a(zW_M^m)\phi_s^T(zW_M^m)$. For $1 \le n \le L_a$ and $1 \le n_1 \le L_s$, the (n, n_1) element of matrix $\Psi(e^{j\omega})$ is given as

$$\begin{aligned} \Psi_{n,n_1}(e^{j\omega}) &= \frac{1}{D}\sum_{m=0}^{M-1}(e^{j\omega}W_M^m)^{-(n-1)}(e^{j\omega}W_M^m)^{-(n_1-1)} \\ &= \begin{cases} 0, & \text{if } \mathrm{mod}(n+n_1-2, M) \neq 0 \\ \frac{M}{D}e^{-j\omega(n+n_1-2)}, & \text{if } \mathrm{mod}(n+n_1-2, M) = 0, \end{cases} \end{aligned} \tag{23}$$

where "mod" denotes the modulus after the division.

In the following, we will formulate the constraints for the total filter bank transfer function. Let $A_T(\omega)$, $\theta_T(\omega)$ and $\tau_T(\omega)$ denote, respectively, the magnitude response, the phase and the group delay of the total filter bank with the transfer function $T(e^{j\omega})$. Moreover, denote the desired frequency response of the filter bank as

$$T_d(e^{j\omega}) = e^{j\hat{\theta}_T(\omega)},\ \forall\ \omega \in [-\pi, \pi] \tag{24}$$

where $\hat{\theta}_T(\omega) = -\omega\hat{\tau}_T$ and $\hat{\tau}_T$ is the desired group delay. The total response of the filter bank is constrained to the desired response according to

$$|T(e^{j\omega}) - T_d(e^{j\omega})| \le \varepsilon_T,\ \forall\ \omega \in [-\pi, \pi] \tag{25}$$

where ε_T is a small specified error. By extending the group delay error approximation for the analysis filter bank in Section 3 to the total analysis and synthesis filter bank, the group delay error for the total filter bank at each frequency ω,

$$e_T(\omega) \quad = \quad \hat{\tau}_T - \tau_T(\omega),\ \forall\ \omega \in [-\pi, \pi]. \tag{26}$$

can be approximated as

$$e_T(\omega) \approx I\left\{T_1(e^{j\omega})e^{-j\hat{\theta}_T(\omega)} + jT(e^{j\omega})e^{-j\hat{\theta}_T(\omega)}\hat{\tau}_T\right\} \tag{27}$$

where

$$T_1(e^{j\omega}) = \mathbf{h}^T \Xi(\omega)\mathbf{g} \tag{28}$$

and $\Xi(\omega)$ is an $L_a \times L_s$ matrix. For $1 \le n \le L_a$ and $1 \le n_1 \le L_s$, the (n, n_1) element of $\Xi(\omega)$ is given as:

$$\Xi_{n,n_1}(\omega) = \begin{cases} 0, & \text{if } \mathrm{mod}(n+n_1-2, M) \neq 0 \\ \frac{-jM(n+n_1-2)}{D}e^{-j\omega(n+n_1-2)}, & \text{otherwise.} \end{cases} \tag{29}$$

Thus, the group delay error can be approximated as a linear function of the synthesis prototype filter $\mathbf{g}$,

$$e_T(\omega) \approx E_T(\omega)\mathbf{g} \tag{30}$$

where

$$E_T(\omega) = \mathbf{h}^T I\left\{\Xi(\omega)e^{-j\hat{\theta}_T(\omega)} + j\Psi(e^{j\omega})e^{-j\hat{\theta}_T(\omega)}\hat{\tau}_T\right\}. \tag{31}$$

For $1 \le d \le D-1$, the second term in (7) can be viewed as the transfer function which contributes to the aliasing terms in the output signal. Thus, the objective is to minimize the total residual aliasing energy for all the frequencies $\omega \in [-\pi, \pi]$, given by

$$\gamma(\mathbf{g}) = \frac{1}{2\pi D^2}\int_{-\pi}^{\pi}\sum_{d=1}^{D-1}\sum_{m=0}^{M-1}|\mathbf{h}^T \Phi_{m,d}(e^{j\omega})\mathbf{g}|^2 d\omega \tag{32}$$

where

$$\Phi_{m,d}(e^{j\omega}) = \phi_a(e^{j\omega}W_M^m W_D^d)\phi_s^T(e^{j\omega}W_M^m) \tag{33}$$

and $(\cdot)^T$ denotes the transpose operation of a vector. This aliasing can be reduced to a quadratic function of $\mathbf{g}$ as follows

$$\gamma(\mathbf{g}) = \mathbf{g}^T \mathbf{P}_a(\mathbf{h})\mathbf{g} \tag{34}$$

where

$$\mathbf{P}_a(\mathbf{h}) = \frac{1}{2\pi D^2}\sum_{d=1}^{D-1}\sum_{m=0}^{M-1}\int_{-\pi}^{\pi}\Phi_{m,d}^H(e^{j\omega})\mathbf{h}^*\mathbf{h}^T \Phi_{m,d}(e^{j\omega})d\omega \tag{35}$$

and $(\cdot)^*$ denote the conjugate operations of a vector.

The design of the synthesis filter bank can be posed as: *Minimize the residual aliasing (32) subject to constraints on the total response of the filter bank (25) and constraints on the group delay error (30).* This optimization problem can be formulated in terms of the synthesis prototype filter as follows,

$$\begin{cases} \min_{\mathbf{g}} \mathbf{g}^T \mathbf{P}_a(\mathbf{h})\mathbf{g} \\ |\mathbf{h}^T \Psi(e^{j\omega})\mathbf{g} - e^{-j\omega\hat{\tau}_T}| \le \varepsilon_T, & \forall\ \omega \in [-\pi, \pi] \\ |E_T(\omega)\mathbf{g}| \le \varepsilon_{T,\tau}, & \forall\ \omega \in [-\pi, \pi]. \end{cases} \tag{36}$$

By ensuring that $|E_T(\omega)\mathbf{g}| \le \varepsilon_{T,\tau}$ for all ω, the group delay error is constrained for all the frequencies within a maximum error of $\varepsilon_{T,\tau}$. Using the real rotation theorem [9], (36) can be reduced to a semi-infinite quadratic optimization problem, which can be solved efficiently by using the discretization method.

5. DESIGN EXAMPLE

Consider the design of a filter bank with $M = 16$ subbands. The decimation factor is chosen as $D = M/2$. The length of the analysis and synthesis prototype filters is $L_a = L_s = 4M$. The desired delay for the total filter bank equals to half of the length of the prototype filter, $\tau_T = L_a/2$. The delay for the analysis prototype filter is $\tau_H = \tau_T/2$.

The analysis filter bank is designed according to (21) with minimum inband aliasing subject to frequency response constraints and linear phase approximation in the passband. Fig. 2 plots the frequency response of the analysis prototype filter with the stopband frequency $\omega_p = \pi/M$. The constraint for the passband is $\varepsilon_H = 0.01$ while the group delay constraint for the analysis filter bank is $\varepsilon_{H,\tau} = 0.01$. It can be seen from the figure that the group delay error approximation for the analysis prototype filter is highly accurate with the group delay error less than $\varepsilon_{H,\tau}$.

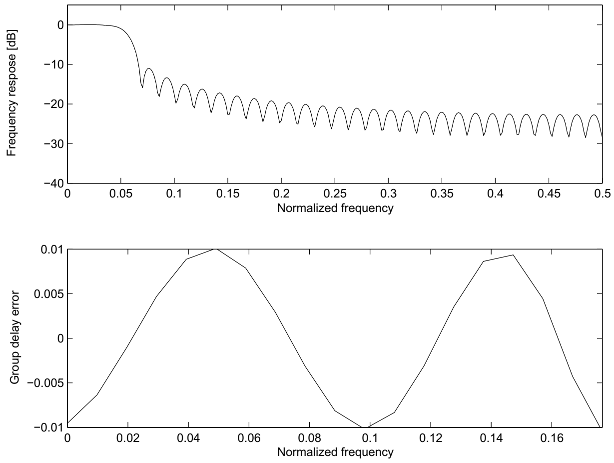

Figure 2: Frequency response and group delay error of the analysis prototype filter.

The constraint on the group delay error for the total filter bank is reduced to a smaller error $\varepsilon_{T,\tau} = 0.001$ while the constraints on the transfer function of the filter bank is $\varepsilon_T = 0.01$. The synthesis filter bank is then designed according to (36). The frequency response error and the total group delay error is plotted in Fig. 3. It can be seen that the group delay error approximation is highly accurate with the group delay error smaller than a desired value at all the frequencies. Thus, by using the proposed design method, the analysis and synthesis prototype filters can be designed accordingly using multiple criteria depending on the practical applications.

6. CONCLUSIONS

In this paper, a formulation for the design of filter bank is proposed for the analysis and synthesis prototype filter with multiple criteria. Accurate approximations for the group delay errors are derived to allow the analysis filter bank having approximately linear phase in the passband while the total filter bank having approximately linear phase at all the frequency components. By employing these approximations, the multi-criteria optimization problems can be reduced to quadratic optimization problems. A design example shows

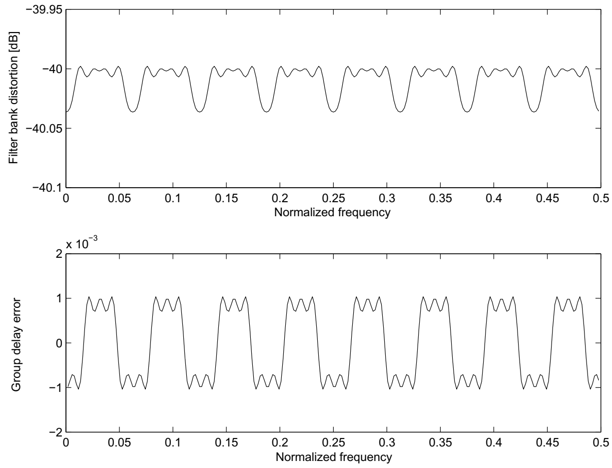

Figure 3: Distortion and group delay error for the total filter bank.

that the group delay error approximations are highly accurate with the group delays restricted to small errors.

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