GENERALIZED VECTOR MEDIANS FOR CORRELATED CHANNELS

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ABSTRACT

Inspired by the maximum likelihood (ML) estimates of location in multivariate spaces, we introduce in this paper a new filtering structure capable of capturing and exploiting both spatial and cross-channel correlations embedded in the data. An adaptive optimization algorithm for a sub-optimal realization of the proposed generalized vector median (GVM) filter, namely the marginal GVM, is derived. The effectiveness of the algorithm is shown through a color image denoising experiment.

1. INTRODUCTION

The vector median (VM) proposed by Astola *et. al* [1] in 1990 has received considerable attention in signal processing research. Their approach is based on the concept of reduced ordering [2], overcoming the limitation (inexistence) of natural ordering in a multivariate space. The vector median of N multivariate samples $X_1, X_2, \cdots, X_N$, thus, computes the minimum sum of L_1 distances from the output candidate to all other multivariate samples in the observation window. Since the minimum of the above cost function cannot be computed easily but by searching or approximations [1], a simple approach is to define the output of the vector median as one of the input vectors which minimizes the sum of distances. The output can thus be written as

$$Y = \arg\min_{X \in \{X_i\}} \sum_{i=1}^{N} \|X - X_i\|_1 \qquad (1)$$

where $\|\cdot\|_1$ denotes the L_1 norm. In order to expand the capabilities of the vector median, the weighted vector median (WVM) was introduced as a direct extension [4]. Here, a weighted cost function is defined where the L_1 distances are first weighted by a non-negative scalar before they are summed to form the cost function

$$Y = \arg\min_{X \in \{X_i\}} \sum_{i=1}^{N} W_i \|X - X_i\|_1 \ . \qquad (2)$$

Although the WVM in (2) can be further extended to a more general form that accepts negative weights as well as positive ones with "sign coupling" weighting [3], the principles of parameter estimation reveal that the limitations of the weighted vector median reside in its very structure. We show that Astola's vector median is derived from the maximum likelihood (ML) estimation of location for independent and identically distributed (i.i.d.) vector-valued samples obeying a Laplacian distribution. The weighted vector median, in turn, emerges from the location estimate of independent (but not identically distributed) vector valued samples, where only the scale of each input vector sample varies. The multi-channel components of each sample are, however, still considered mutually independent in both cases. Weighted vector medians, as defined in (2), are thus severely limited as the cross-channel correlation structure, inherently present in most multi-channel applications, cannot be exploited. In short, the vector median in [1] is spatially blind and cross-channel blind. Similarly, the weighted vector median in [4] is cross-channel blind.

This paper focuses on developing more general multivariate median filter structures that are capable of capturing and exploiting spatial and cross-channel correlations embedded in the data. We start by revisiting the multivariate location estimate of samples that are assumed to be mutually correlated across channels but that are assumed independent (but not identical) in time/space. This model leads to a multivariate median structure that is computationally simple, yet it exploits cross-channel information. The structure is also amenable to "sign coupling" weighting and thus can be adapted to admit positive and negative weights.

Having the new generalized vector median (GVM) filter structure defined in Section 3, our attention then focuses on the weight optimization. In Section 4, the adaptive algorithm for the marginal GVM filter is presented. To verify the validity of the algorithm, a color image denoising example with salt and pepper noise is introduced in Section 5, where the advantages of the newly proposed GVM method are shown.

2. STATISTICAL FOUNDATION

Filtering and parameter estimation are intimately related. It is well known that, in the univariate case, the optimal linear filter and the optimal median filter arise from the maximum likelihood estimate of location under Gaussian and Laplacian statistics, respectively. Consider a set of independent, univariate samples $\{X_i\}$ each obeying a Gaussian distribution but with a different variance σ_i^2. The ML estimate of location in this case is equivalent to the minimizer of the following function

$$G_2(\mu) = \sum_{i=1}^{N} \frac{1}{\sigma_i^2}(X_i - \mu)^2,$$

known as the normalized weighted mean

$$\bar{\mu} = \frac{\sum_{i=1}^{N} W_i X_i}{\sum_{i=1}^{N} W_i}$$

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with $W_i = \sigma_i^2 > 0$, which is a normalized version of a FIR filter

$$Y = \sum_{i=1}^{N} W_i X_i \ .$$

A similar link between filtering and maximum likelihood (ML) estimation can be found when the univariate samples obey independent but not identical Laplacian distributions. The ML estimate of location now minimizes

$$G_1(\mu) = \sum_{i=1}^{N} \frac{1}{\sigma_i^2} |X_i - \mu|,$$

which can be calculated by the popular nonlinear operation known as the weighted median

$$\tilde{\mu} = \mathrm{MED}(W_i \diamond X_i|_{i=1}^{N})$$

with the limitation of $W_i > 0$. The generalized version which has band-pass and high-pass capabilities by accepting negative weights is further formulated in [3] as

$$Y = \mathrm{MED}(|W_i| \diamond \mathrm{sgn}(W_i) X_i|_{i=1}^{N}),$$

where $\mathrm{sgn}(x) = 1$ when $x \geq 0$ and $\mathrm{sgn}(x) = 0$ otherwise.

It is shown next that, a similar relationship between the ML estimation and filtering can also be set up in the multivariate case.

3. NEW MULTIVARIATE FILTERING STRUCTURE

The notation used hereafter is clarified first. The letter M represents the dimension of the multivariate data, N the filter length, and L the sample size. The filter input vector is denoted as $\mathbf{X} = [X_1\ X_2\ \cdots\ X_N]^T$, where $X_i = [X_i^1\ X_i^2\ \cdots\ X_i^M]^T$ is the i-th M-variate sample in the filter window. The filter output is $Y = [Y^1\ Y^2\ \cdots\ Y^M]^T$. Throughout the paper, only the vectors in time (or spatial) domain are denoted using boldface letters, vectors in the spectral domain are represented in regular font. Also, vectors from two domains do not allow traditional vector operations such as addition, and inner product.

Consider a set of independent but not identically distributed samples, each obeying a joint Gaussian distribution with the same vector location parameter μ,

$$f(X_i) = \frac{1}{(2\pi)^{\frac{M}{2}} |C_i|^{\frac{1}{2}}}\ e^{-\frac{1}{2}(X_i-\mu)^T C_i^{-1}(X_i-\mu)},$$

where X_i and μ are all M-variate column vectors, and C_i is the cross-channel $M \times M$ correlation matrix of the sample X_i. The maximum likelihood estimate of location μ can be derived as

$$\bar{\mu} = \left(\sum_{i=1}^{N} C_i^{-1}\right)^{-1} \left(\sum_{i=1}^{N} C_i^{-1} X_i\right).$$

Just like in the univariate case, a general filtering structure for multivariate data can thus be inspired from the above ML estimate of location

$$Y = \sum_{i=1}^{N} W_i^T X_i$$

where $W_i = (W_i^{jl})_{M\times M}$ and there is a total of N different weight matrices. A corresponding optimization algorithm for this filtering structure can be easily developed, but its shortcoming is obvious: the number of weights could be unbearably large. For instance, for a 3-channel color image with a window size 5x5, we need 225 weights to perform the filtering operation.

Fortunately, in many multi-channel applications such as color imaging, remote sensing, array processing, etc., the signals from sub-channels are often highly correlated. Hence, it is possible to utilize this cross-channel correlation structure to reduce filter complexity. Moreover, this correlation structure between sub-channels may often be stationary or at least quasi-stationary for a period of time. In these cases, it can be assumed that the correlation matrices C_i within the observation window differ only by scale factors. This relationship can be stated as

$$C_i^{-1} = q_i C^{-1},$$

where the scalars $q_i > 0$ for $i = 1, 2, \cdots, N$. Then, the corresponding ML estimate of location is

$$\bar{\mu} = \left(\sum_{i=1}^{N} q_i C^{-1}\right)^{-1} \left(\sum_{i=1}^{N} q_i C^{-1} X_i,\right)$$

where $\left(\sum_{i=1}^{N} q_i C^{-1}\right)^{-1}$ is a normalizing matrix and $\left(\sum_{i=1}^{N} q_i C^{-1} X_i\right)$ is the inherent filter structure. Note that though the correlation matrix C in the ML estimation for μ can be cancelled out, the essence of the multivariate filtering resides in $\left(\sum_{i=1}^{N} q_i C^{-1} X_i\right)$ which leads us to the new filtering structure formulated as

$$\begin{aligned} Y &= \sum_{i=1}^{N} V_i W^T X_i & (3) \\ &= \sum_{i=1}^{N} V_i \begin{bmatrix} \sum_{j=1}^{M} W^{j1} X_i^j \\ \vdots \\ \sum_{j=1}^{M} W^{jM} X_i^j \end{bmatrix}, & (4) \end{aligned}$$

where the weights V_i and W^{jl} all take on real values. For obvious reasons, $\mathbf{V} = [V_1\ V_2\ \cdots\ V_N]^T$ is referred to as the time/spatial dependent weight vector, and $W = (W^{jl})_{M\times M}$ as the cross-channel weight matrix.

Though it is mathematically intractable to derive a similar result as in (3) from a multivariate Laplacian distribution since it involves special functions such as Bessel and Gamma, by analogy, we define a powerful nonlinear multivariate filter by first replacing the summations in (4) with median operators and then incorporating the "sign coupling" accordingly. The newly formulated filter is referred to as the generalized vector median (GVM) shown as follows

$$Y = \mathrm{MED}(|V_i| \diamond \mathrm{sgn}(V_i) Q_i\ |_{i=1}^{N}), \qquad (5)$$

where

$$Q_i = \begin{bmatrix} \mathrm{MED}(|W^{j1}| \diamond \mathrm{sgn}(W^{j1}) X_i^j\ |_{j=1}^{M}) \\ \vdots \\ \mathrm{MED}(|W^{jM}| \diamond \mathrm{sgn}(W^{jM}) X_i^j\ |_{j=1}^{M}) \end{bmatrix}$$

is an M-variate vector. It is clear to see that, if we set W to be the identity matrix and confine V_i to be positive only, then the GVM filter in (5) will reduce to the traditional weighted vector median which is cross-channel blind. Moreover, if we further set all V_i to be the same value, the filter then becomes Astola's VM which is time/spatial blind as well.

Since the median of a set of vectors is not uniquely defined [2], the GVM filter in (5) may have different interpretations and implementations. Moreover, among all the approaches available in the literature [6], none of them has a closed form formula, which makes the filter optimization cumbersome. To circumvent the obstacle of optimizing this filter, we propose a sub-optimal implementation referred to as the marginal GVM, where the outer median in (5) is replaced by a vector of marginal medians each with the same weight vector $\mathbf{V}$

$$Y = \begin{bmatrix} \mathrm{MED}(|V_i| \diamond \mathrm{sgn}(V_i) Q_i^1 \mid_{i=1}^N) \\ \mathrm{MED}(|V_i| \diamond \mathrm{sgn}(V_i) Q_i^2 \mid_{i=1}^N) \\ \vdots \\ \mathrm{MED}(|V_i| \diamond \mathrm{sgn}(V_i) Q_i^M \mid_{i=1}^N) \end{bmatrix}, \tag{6}$$

where $Q_i^l = \mathrm{MED}(|W^{jl}| \diamond \mathrm{sgn}(W^{jl}) X_i^j \mid_{j=1}^M)$ for $l = 1, \cdots, M$. Now using this structure to filter a color image with a 5×5 window, we only need 34 weights.

4. FILTER OPTIMIZATION

Assume that the observed process $X(n)$ is statistically related to a desired process $D(n)$ of interest, that is typically considered a transformed or corrupted version of $D(n)$. The filter input vector at time n is

$$\mathbf{X}(n) = [X_1(n)\ X_2(n)\ \cdots\ X_N(n)]^T,$$

where $X_i(n) = [X_i^1(n)\ X_i^2(n)\ \cdots\ X_i^M(n)]^T$ is an M-variate sample. The desired signal at time n is $D(n) = [D^1(n)\ D^2(n)\ \cdots\ D^M(n)]^T$. From (6), the output of the marginal generalized vector median filter is denoted as $\hat{D} = [\hat{D}^1\ \hat{D}^2\ \cdots\ \hat{D}^M]^T$ where

$$\hat{D}^l = \mathrm{MED}(|V_i| \diamond \mathrm{sgn}(V_i) Q_i^l \mid_{i=1}^N) \quad l = 1, \cdots, M. \tag{7}$$

Applying the real-valued threshold decomposition technique [3], we can rewrite (7) making it an analyzable function as follows,

$$\hat{D}^l = \frac{1}{2} \int \mathrm{sgn}(\mathbf{V}_a^T \mathbf{G}^{p^l}) dp^l,$$

where $\mathbf{V}_a = [|V_1|\ |V_2|\ \cdots\ |V_N|]^T$, and

$$\mathbf{G}^{p^l} = [\mathrm{sgn}(\mathrm{sgn}(V_1) Q_1^l - p^l)\ \cdots\ \mathrm{sgn}(\mathrm{sgn}(V_N) Q_N^l - p^l)]^T.$$

Similarly, by defining $W_a^l = [|W^{1l}|\ |W^{2l}|\ \cdots\ |W^{Ml}|]^T$, and

$$S_i^{q_i^l} = [\mathrm{sgn}(\mathrm{sgn}(W^{1l}) X_i^1 - q_i^l)\ \cdots\ \mathrm{sgn}(\mathrm{sgn}(W^{Ml}) X_i^M - q_i^l)]^T,$$

the inner weighted medians will have the following thresholded representations

$$Q_i^l = \frac{1}{2} \int \mathrm{sgn}((W_a^l)^T S_i^{q_i^l}) dq_i^l.$$

Under the Mean L_1-norm (ML1) criterion, the cost function to be minimized is

$$J_1(\mathbf{V}, W) = E\{\|D - \hat{D}\|_1\} = E\left\{\sum_{l=1}^M |D^l - \hat{D}^l|\right\}. \tag{8}$$

Following similar arguments as in [3], the derivative of (8) with respect to V_i can be approximated by the following summation

$$\frac{1}{2} \sum_{l=1}^M \int E\left\{e^{p^l} \mathrm{sech}^2(\mathbf{V}_a^T \mathbf{G}^{p^l}) \mathrm{sgn}(V_i) G_i^{p^l}\right\} dp^l,$$

where $e^{p^l} = \mathrm{sgn}(D^l - p^l) - \mathrm{sgn}(\mathbf{V}_a^T \mathbf{G}^{p^l})$, and $G_i^{p^l} = \mathrm{sgn}(\mathrm{sgn}(V_i) Q_i^l - p^l)$ for $i = 1, \cdots, N$.

Using the instantaneous estimate for the gradient, and applying the approximation technique used in [3], we obtain the adaptive algorithm for the time dependent weight vector $\mathbf{V}$ in the marginal GVM filter as follows,

$$V_i(n+1) = V_i(n) + \mu_v \mathrm{sgn}(V_i(n)) e^T(n) G_i^{\hat{D}}(n), \tag{9}$$

where $G_i^{\hat{D}} = [G_i^{\hat{D}^1}\ \cdots\ G_i^{\hat{D}^M}]^T$ and $G_i^{\hat{D}^l} = \mathrm{sgn}(\mathrm{sgn}(V_i) Q_i^l - \hat{D}^l)$ for $l = 1, \cdots, M$.

After some mathematical manipulations and approximations, the adaptive algorithm for the cross-channel weight matrix W in the marginal GVM filter can be simplified as follows

$$W^{st}(n+1) = W^{st}(n) + \mu_w \mathrm{sgn}(W^{st}(n)) e^t(n) (\mathbf{V}^T(n) \mathbf{A}^s(n)), \tag{10}$$

where $\mathbf{A} = [A_1^s\ A_2^s\ \cdots\ A_N^s]^T$, $A_i^s = \delta(\mathrm{sgn}(V_i) Q_i^l - \hat{D}^l) \mathrm{sgn}(\mathrm{sgn}(W^{st}) X_i^s - Q_i^t)$ for $i = 1, \cdots, N$, and

$$\delta(x) = \begin{cases} 1 & x = 0 \\ 0 & \text{otherwise} \end{cases}.$$

$\mathbf{V}$ should be initialized as an all-1 vector in marginal GVM.

5. SIMULATIONS

To test the performance of the GVM filter proposed in this paper, the following experiment is executed.

A RGB color image contaminated with 10% correlated salt and pepper noise is processed by the WVM filter, and the marginal GVM filter separately. The observation window is set to 3×3 and 5×5. The optimal weights for the marginal GVM filter are first obtained by running the LML1 algorithm derived in the previous section over a small part of the corrupted image. The corresponding part of the clean image is considered known to the algorithm. A similar procedure is repeated when the Algorithm I developed in [5] is used to optimize the weights of the WVM filter. The adaptation parameters are chosen in a way such that the average L_1-norm error obtained in the training process is close to its minimum for each filter. The resulting weights are then passed to the corresponding filters to denoise the whole image. The filter outputs are depicted in Figure 1.

As a measure of the effectiveness of the filters, the average L_1-norm error of the outputs was calculated for each filter, the results are summarized in Table 1. Peak signal-to-noise ratio (PSNR) was also used to evaluate the fidelity of the two filtered images.

Figure 1: Multivariate medians for color images in salt and pepper noise, $\mu = 0.001$ for WVM, $\mu_v, \mu_w = 0.05$ for marginal GVM. From left to right and top to bottom: noiseless image, contaminated image, marginal GVM with 3x3 window, WVM with 3x3 window, marginal GVM 3x3, WVM 5x5.

The statistics in Table 1 show that the marginal GVM filter outperforms the WVM filter in this color image denoising simulation by a factor of 3 in terms of the average L_1-norm error, or 8-11dB in terms of PSNR. Moreover, the output of the marginal GVM filter is almost salt and pepper noise free. As a comparison, the output of the WVM filter is visually less pleasant with many unfiltered outliers. Notice that the output of the marginal GVM filter with the 3x3 observation window preserves more image details than that of the 5x5 realization, and has a better PSNR though the average L_1-norm errors in the two cases are roughly the same.

Table 1: Average ML1 and PSNR of the output images.

Filter	ML1		PSNR (dB)	
	3×3	5×5	3×3	5×5
Noisy signal	0.1506		14.04	
WVM	0.0748	0,0732	21.75	23.33
marginal GVM	0.0248	0.0247	32.60	30.61

6. CONCLUSIONS

A novel multivariate median filter was proposed for color image applications. The corresponding optimal filter design was derived. The simulation on color image denoising shows the superiority of the new structure over the traditional weighted vector median filter.

REFERENCES

[1] J. Astola, J. Haavisto, and Y. Neuvo, "Vector median filters," *Proc. IEEE*, vol. 78, pp. 678–689, Apr. 1990.

[2] V. Barnett, "The ordering of multivariate data," *Journal of the Royal Statistical Soceity, Series A (General)*, vol. 139, no. 3, pp. 318–355, Apr. 1976.

[3] G. R. Arce, "A general weighted median filter structure admitting negative weights," *IEEE Transactions on Signal Processing*, vol. 46, pp. 3195–3205, Dec. 1998.

[4] T. Vireo, K. Öistämö, and Y. Neuvo, "Three-dimensional median-related filters for color image sequence filtering," *IEEE Trans. Circuits and Systems for Video Technology*, vol. 4, pp. 129–142, Apr. 1994.

[5] Y. Shen and K. E. Barner, "Fast optimization of weighted vector median filters," *Proc. ICASSP04*, May 2004.

[6] G. Aloupis, *On computing geometric estimators of location*. M.Sc. Thesis, McGill University, 2001.