STATISTICAL ANALYSIS OF THE KUMARESAN–TUFTS AND MATRIX PENCIL METHODS IN ESTIMATING A DAMPED SINUSOID

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ABSTRACT

Several methods have been developed for estimating the parameters of damped and undamped exponentials in noise, but the performances of such techniques are generally known only in the undamped case. In this paper, we consider two estimation methods: the Kumaresan–Tufts method and the Matrix Pencil approach, and we obtain their estimation performances in the case of a single exponentially damped sinusoid. Assuming a high signal-to-noise ratio, closed form expressions for the bias and the variance of the damping factor are derived. The analytical results are confirmed using Monte Carlo simulations. The analysis indicates that the Matrix Pencil method exhibits a lower variance but has a greater bias than the Kumaresan–Tufts approach.

1. INTRODUCTION

There are several methods for the estimation of the parameters of damped or undamped exponential signals in noise. In the context of damped complex exponentials, the most popular parametric estimation method is the well-known Kumaresan and Tufts (KT) approach [1]. It performs a reduced rank pseudoinverse of the data matrix to get backward linear prediction parameters, from which the signal modes may be obtained. Another estimation method is the Matrix Pencil (MP) method which is based on a matrix prediction equation [2]. The performances of these methods and others are well-known in the case of undamped sinusoids [2, 3], but for damped exponentials only few works are available [4, 5], and what is more the bias is assumed to be zero. Of course, this assumption is valid for the frequency estimate at a high signal–to–noise ratio (SNR) but, as will be shown later, it does not hold for the damping factor.

In this paper, we extend these previous works to take into account the bias in the damping factor, assuming a high SNR. It is obtained by considering the first order perturbation in the highest singular value of the data matrix. We show that the frequency estimate obtained by both KT and MP methods are unbiased and that the damping factor is biased. We also derive the expressions of the mean and the variance of the damping factor for both methods.

The paper is organized as follows. In section 2, a brief recall about the two estimation methods considered here (i.e. KT and MP) is presented. Section 3 is devoted to the performance analysis of both methods using Wilkinson's approach [6]. In particular, the expressions for the first and second order moments will be given. In section 4, a simulation example is then presented in order to compare the theoretical results with the experimental ones. Finally, the conclusions are given in section 5.

2. BACKGROUND OF THE ESTIMATION METHODS

Consider the following complex signal composed of one damped exponential in an additive Gaussian white noise:

$$\tilde{x}(n) = x(n) + e(n) = hz^n + e(n),\ n = 0, ..., N-1 \quad (1)$$

where $z = \exp(\alpha + j\omega)$ is the damped mode ($\alpha < 0$) with complex amplitude h. The noise $e(n)$ is zero mean with variance σ_e^2. Throughout this paper, the tilde symbol $(\tilde{\cdot})$ indicates a noisy variable. The methods described below estimate the signal mode z using linear prediction and singular value decomposition (SVD).

2.1 The KT method

The KT method is a backward linear prediction technique. It consists first in forming the following equation system using the available data:

$$\tilde{X}_1 \tilde{\boldsymbol{b}} = -\tilde{\boldsymbol{x}}_0$$

where

$$\tilde{X}_1 = \begin{pmatrix} \tilde{x}(1) & \cdots & \tilde{x}(p) \\ \tilde{x}(2) & \cdots & \tilde{x}(p+1) \\ \vdots & & \vdots \\ \tilde{x}(N-p) & \cdots & \tilde{x}(N-1) \end{pmatrix},$$

$$\tilde{\boldsymbol{b}} = [\tilde{b}_1,\ ...,\ \tilde{b}_p]^T,$$

$$\tilde{\boldsymbol{x}}_0 = [\tilde{x}(0),\ ...,\ \tilde{x}(N-p-1)]^T.$$

and p is the prediction order. The backward prediction coefficients are obtained by performing a reduced rank pseudo–inverse of the data matrix:

$$\tilde{\boldsymbol{b}} = -\tilde{X}_1^{\dagger}\tilde{\boldsymbol{x}}_0 = -\frac{1}{\tilde{\sigma}_1}\tilde{\boldsymbol{v}}_1\tilde{\boldsymbol{u}}_1^H\tilde{\boldsymbol{x}}_0 \quad (2)$$

where $\tilde{\sigma}_1$ is the principal singular value of the data matrix $\tilde{X}_1$, and $\tilde{\boldsymbol{u}}_1$ and $\tilde{\boldsymbol{v}}_1$ are the principal left and right singular vectors, respectively. Once the prediction coefficients are obtained, the roots of the polynomial:

$$\tilde{B}(z) = 1 + \sum_{i=1}^{p} \tilde{b}_i z^{-i} = \prod_{i=0}^{p-1}(1 - \tilde{z}_i z^{-1})$$

are computed and the one which lies outside the unit circle, denoted by $\tilde{z}_0$, is selected. It corresponds to the inverse of the signal mode:

$$\tilde{z}_0 = 1/\tilde{z} \quad (3)$$

2.2 The MP method

In the case of the MP method [2], we form two matrices $\tilde{X}_0$ and $\tilde{X}_1$. The matrix $\tilde{X}_1$ is the same as in the KT method, and $\tilde{X}_0$ is obtained in the same manner, i.e.

$$\tilde{X}_0 = \begin{pmatrix} \tilde{x}(0) & \cdots & \tilde{x}(p-1) \\ \tilde{x}(1) & \cdots & \tilde{x}(p) \\ \vdots & & \vdots \\ \tilde{x}(N-p-1) & \cdots & \tilde{x}(N-2) \end{pmatrix}$$

Then, using the reduced rank pseudo–inverse of the matrix $\tilde{X}_1$, the matrix $\tilde{Z}$ is computed as follows:

$$\tilde{Z} = \tilde{X}_1^{\dagger}\tilde{X}_0 = \frac{1}{\tilde{\sigma}_1}\tilde{\boldsymbol{v}}_1\tilde{\boldsymbol{u}}_1^H\tilde{X}_0 \tag{4}$$

The eigenvalue $\tilde{z}_0$ of the matrix $\tilde{Z}$ lying outside the unit circle (i.e. $|\tilde{z}_0| > 1$) is selected. As for the KT method, it is related to the signal mode by relation (3).

3. STATISTICAL ANALYSIS

In the noiseless case, it can be easily shown (e.g. [5]) that:

$$\boldsymbol{v}_1 = \frac{z^*}{|z|\sqrt{k_v}}\begin{pmatrix} 1 \\ z^* \\ \vdots \\ z^{*p-1} \end{pmatrix}, \ \boldsymbol{u}_1 = \frac{h}{|h|\sqrt{k_u}}\begin{pmatrix} 1 \\ z \\ \vdots \\ z^{N-p-1} \end{pmatrix}$$

where

$$k_v = \sum_{i=0}^{p-1}\exp(2\alpha i), \quad k_u = \sum_{i=0}^{N-p-1}\exp(2\alpha i)$$

In addition, the nonzero singular value and the prediction vector are:

$$\sigma_1 = |h||z|\sqrt{k_v k_u}, \quad \boldsymbol{b} = \frac{-1}{|z|\sqrt{k_v}}\boldsymbol{v}_1$$

In the forthcoming analysis, we first consider the first order perturbation in the "dominant" singular value σ_1 and we give its mean and variance. Then, we describe the statistical analysis of the mode estimates obtained by the KT and MP methods. Throughout this paper, $z_0 \triangleq 1/z = \exp(-\alpha - j\omega)$.

3.1 Perturbation in the signal singular value

The signal-related singular value is the square root of the unique nonzero eigenvalue of the matrix $X_1^H X_1$ (or $X_1 X_1^H$). In the noisy case, we have:

$$\tilde{X}_1^H\tilde{X}_1 = X_1^H X_1 + \underbrace{(E_1^H E_1 + E_1^H X_1 + X_1^H E_1)}_{\text{error term}}$$

where E_1 is a Hankel matrix of noise entries: $\tilde{X}_1 = X_1 + E_1$. It is known from perturbation theory that the first order perturbation (as $\sigma_e^2 \to 0$) in the eigenvalue σ_1^2 of the matrix $X_1^H X_1$, associated to the eigenvector $\boldsymbol{v}_1$, is given by $\Delta\sigma_1^2 = \tilde{\sigma}_1^2 - \sigma_1^2$, where [6]:

$$\Delta\sigma_1^2 = \boldsymbol{v}_1^H(E_1^H E_1 + E_1^H X_1 + X_1^H E_1)\boldsymbol{v}_1$$

From this equation, it is clear that the mean of $\Delta\sigma_1^2$ is

$$\mathbb{E}\left\{\Delta\sigma_1^2\right\} = (N-p)\sigma_e^2$$

For the remaining of the analysis, we need also the variance of $\Delta\sigma_1^2$. Its calculation is not especially difficult, but it is extremely long and it is not possible to reproduce it here. We found it to be [7]

$$\mathrm{var}\{\Delta\sigma_1^2\} = 2|h|^2\sigma_e^2 s_2 + \sigma_e^4\Bigg[-(N-p) + 2\sum_{i=0}^{m-1}(N-p-i)|z|^{2i}\Big(\frac{1-|z|^{2(p-i)}}{1-|z|^{2p}}\Big)^2\Bigg]$$

where

$$s_2 = \sum_{i=0}^{m-1} i^2|z|^{2i} + m^2\sum_{i=m}^{N-m}|z|^{2i} + \sum_{i=N-m+1}^{N-1}(N-i)^2|z|^{2i}$$

and $m = \min(p, N-p)$.

3.2 Analysis of the KT method

At high SNR, the matrix $\tilde{X}_1$ is approximately rank one, so we have:

$$\tilde{X}_1^{\dagger} \simeq \frac{\tilde{X}_1^H}{\tilde{\sigma}_1^2} \tag{5}$$

According to equation (2), one can conclude that:

$$\tilde{\boldsymbol{b}} \simeq -\frac{1}{\tilde{\sigma}_1^2}\tilde{X}_1^H\tilde{\boldsymbol{x}}_0$$

As $\tilde{X}_1 = X_1 + E_1$, $\tilde{\boldsymbol{x}}_0 = \boldsymbol{x}_0 + \boldsymbol{e}_0$ and $1/\tilde{\sigma}_1^2 \simeq (1 - \frac{\Delta\sigma_1^2}{\sigma_1^2})/\sigma_1^2$, the first order perturbation in the prediction coefficients is $\Delta\boldsymbol{b} = \tilde{\boldsymbol{b}} - \boldsymbol{b}$, where

$$\Delta\boldsymbol{b} = -\frac{1}{\sigma_1^2}(X_1^H\boldsymbol{e}_0 + E^H\boldsymbol{x}_0 + \boldsymbol{b}\Delta\sigma_1^2) \tag{6}$$

It is easy to observe that the vector of prediction coefficients is biased since

$$\mathbb{E}\{\Delta\boldsymbol{b}\} = -\frac{\mathbb{E}\left\{\Delta\sigma_1^2\right\}}{\sigma_1^2}\boldsymbol{b} = \frac{(N-p)\sigma_e^2}{\sigma_1^2|z|\sqrt{k_v}}\boldsymbol{v}_1 \tag{7}$$

We have also derived the expression of the covariance matrix of $\Delta\boldsymbol{b}$ using equation (6). Because of the lack of space, we cannot give this expression here.

The error $\Delta\boldsymbol{b}$ in the prediction coefficients results in a deviation of the true root $z_0 = 1/z$ of the polynomial $B(z)$ to a new position $\tilde{z}_0 = z_0 + \Delta z_0$ where [8]:

$$\Delta z_0 = -\sum_{k=1}^{p}\frac{z_0^{p-k}}{\prod_{i=1}^{p-1}(z_0 - z_i)}\Delta b_k \tag{8}$$

where $\{z_i\}_{i=1}^{p-1}$ are the "extraneous" roots of the polynomial $B(z)$ (i.e. roots other than z_0). It is known that these roots are distributed over a circle at well defined frequencies. We have found that:

$$z_i = \mu|z|\exp(j\omega_i)$$

where

$$\mu = \frac{1}{\sqrt[p-1]{k_v}},\ \omega_i = -\omega - \frac{2\pi i}{p}$$

Let

$$\boldsymbol{g}_0 = [z_0^{p-1}\ z_0^{p-1},\ \ldots,\ 1]^H = \frac{|z|\sqrt{k_v}}{z^{*p}}\boldsymbol{v}_1$$

$$\beta_0 = \prod_{i=1}^{p-1}(z_0 - z_i) = \frac{1}{z^{p-1}}\frac{1-(\mu|z|^2)^p}{1-\mu|z|^2}$$

then equation (8) may be written as

$$\Delta z_0 = -\frac{1}{\beta_0}\boldsymbol{g}_0^H \Delta \boldsymbol{b} \tag{9}$$

Now, using Eqs. (9) and (7), we get after simplification:

$$\mathbb{E}\{\Delta z_0\} = -\frac{1-\mu|z|^2}{1-(\mu|z|^2)^p}\frac{(N-p)\sigma_e^2}{\sigma_1^2}z_0$$

This equation shows that the frequency estimate is not biased since $\arg\mathbb{E}\{\tilde{z}_0\} = \arg z_0 = -\omega$. But the damping factor is biased. The expression of the bias is:

$$\mathbb{E}\{\Delta\alpha\} \simeq \frac{1-\mu|z|^2}{1-(\mu|z|^2)^p}\frac{(N-p)\sigma_e^2}{\sigma_1^2},$$

where $\Delta\alpha \simeq -\Delta z_0/z_0$ (obtained using a Taylor approximation). The variance of the damping factor is obtained by noting that $\mathbb{E}\left\{(\Delta\alpha)^2\right\} \simeq \frac{1}{2}|z|^2\mathbb{E}\left\{|\Delta z_0|^2\right\}$, and

$$\mathbb{E}\left\{|\Delta z_0|^2\right\} = k_v\left(\frac{1-\mu|z|^2}{1-(\mu|z|^2)^p}\right)^2 \boldsymbol{v}_1^H\mathbb{E}\left\{\Delta\boldsymbol{b}\Delta\boldsymbol{b}^H\right\}\boldsymbol{v}_1 \tag{10}$$

By replacing the expression of the covariance matrix of $\Delta\boldsymbol{b}$ in equation (10), we finally obtain [7]:

$$\mathbb{E}\left\{(\Delta\alpha)^2\right\} = \frac{1}{2\sigma_1^4}\left(\frac{1-\mu|z|^2}{1-(\mu|z|^2)^p}\right)^2\Big[(N-p)^2\sigma_e^4 + \mathrm{var}\{\Delta\sigma_1^2\} + \sigma_1^2\sigma_e^2|z|^2k_v + |h|^2\sigma_e^2 s_2 - 2|h|^2\sigma_e^2|z|^2k_v r_1\Big]$$

where

$$r_1 = \sum_{i=0}^{m-1} i|z|^{2i} + m\sum_{i=m}^{N-p-1}|z|^{2i}$$

3.3 Analysis of the MP method

Using the same approximation of the inverse of $\tilde{X}_1$ in equation (5), and from equation (4), we obtain

$$\Delta Z = \tilde{Z} - Z \simeq \frac{1}{\sigma_1^2}(-Z\Delta\sigma_1^2 + X_1^H E_0 + E_1^H X_0)$$

Using Wilkinson's approach [6], the first order perturbation in the eigenvalue z_0 of the matrix Z is given by:

$$\Delta z_0 = \frac{1}{\sigma_1^2}\boldsymbol{v}_1^H(-Z\Delta\sigma_1^2 + X_1^H E_0 + E_1^H X_0)\boldsymbol{v}_1 \tag{11}$$

So, in backward prediction, the mean deviation of the MP estimate from the mode z_0 is

$$\mathbb{E}\{\Delta z_0\} = -\frac{(N-p)\sigma_e^2}{\sigma_1^2}z_0$$

Since $\arg\mathbb{E}\{z_0 + \Delta z_0\} = \arg z_0 = -\omega$, the frequency estimate is not biased. Regarding the damping factor, it is biased since:

$$\mathbb{E}\{\Delta\alpha\} \simeq \frac{(N-p)\sigma_e^2}{\sigma_1^2}$$

The variance of the error Δz_0 is obtained using equation (11), from which we derive the following expression for the second order moment of $\Delta\alpha$ [7]:

$$\mathbb{E}\left\{(\Delta\alpha)^2\right\} = \frac{1}{2\sigma_1^4}\Big[(N-p)^2\sigma_e^4 + \mathrm{var}\{\Delta\sigma_1^2\} - |h|^2\sigma_e^2(1-|z|^2)s_2 - 2|h|^2|z|^2\sigma_e^2 s_2'\Big]$$

where

$$s_2' = s_2 + \sum_{i=0}^{m-1} i|z|^{2i} - \sum_{i=N-m}^{N-1}(N-i)|z|^{2i}$$

4. SIMULATION EXAMPLE

We consider a signal containing one damped exponential with parameters $N = 30$, $\alpha = -0.1$, $\omega = 2\pi 0.1$ and $SNR = 40$ dB. The peak SNR is defined as:

$$SNR = 10\log(|h|^2/\sigma_e^2)$$

The theoretical bias and variance determined in this paper are compared to those calculated from 2000 numerical simulations. The results obtained with the KT and MP methods are shown in figures 1 and 2 respectively, for a prediction order varying between 1 and $N-1$. First, we can conclude that the variance of the damping factor estimated with both KT and MP methods are minimized for a prediction order $p = N/3$ or $p = 2N/3$. This corresponds to the well–known optimal parametrization of these methods in terms of the minimization of the variance of the frequency error.

Moreover, we observe that the KT method exhibits a lower bias as compared to the MP method, but the latter presents a much lower variance. In the case of damped exponentials, the bias of the damping factor is a very important measure of performance because it informs about the z–plan localization of the estimated mode. Indeed, if the bias is too high, then the mode may be estimated inside the unit circle even in a backward prediction. In this case, if the unit circle criterion is used to select the signal–related modes, some modes will be missed.

Figures 3 and 4 show the bias and the mean square error versus the SNR, for the KT and MP methods. Here the prediction order is set to $p = 10$ and the other parameters are the same as before. These figures show that the theoretical variance is valid beyond a threshold SNR, which is here approximately 10 dB. We also observe that the theoretical bias is valid from 0 dB. The KT method gives a better z–plan localization of the damped mode since it is estimated outside the unit circle for $SNR > 5$ dB while the MP method works for $SNR > 10$ dB.

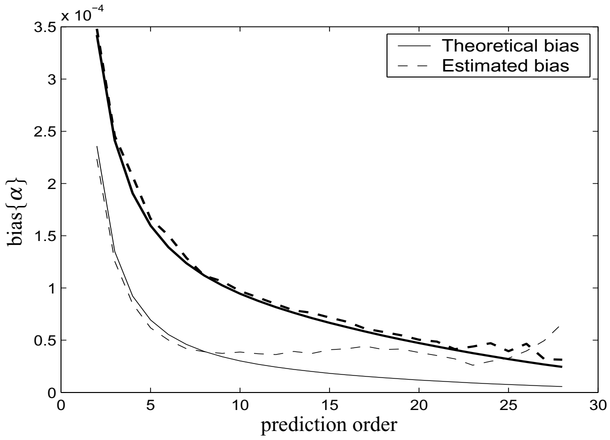

Figure 1: Theoretical and estimated bias of the damping factor for KT (—) and MP (**—**) methods.

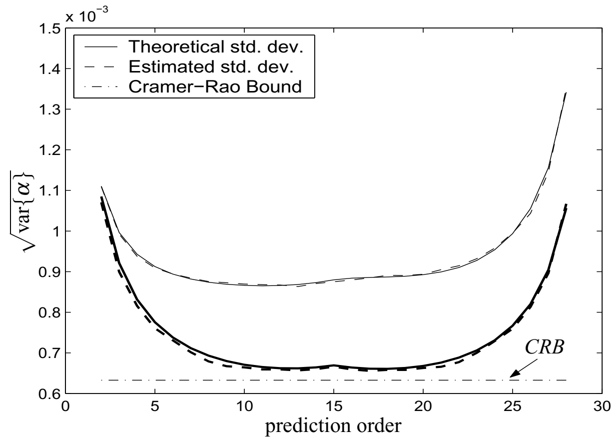

Figure 2: Theoretical and estimated variance of the damping factor for KT (—) and MP (**—**) methods.

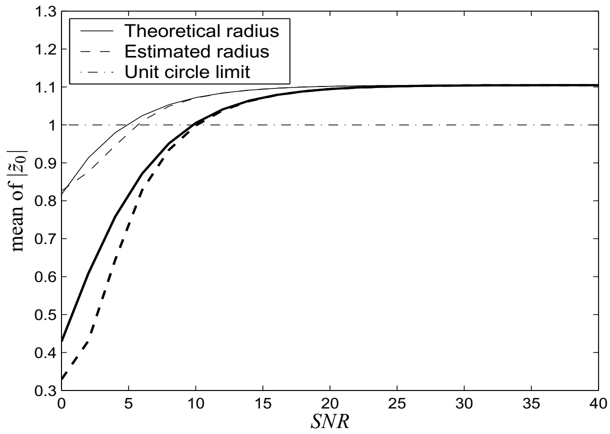

Figure 3: Theoretical and estimated radius of the signal mode in backward prediction as a function of the SNR. (—) KT method; (**—**) MP method.

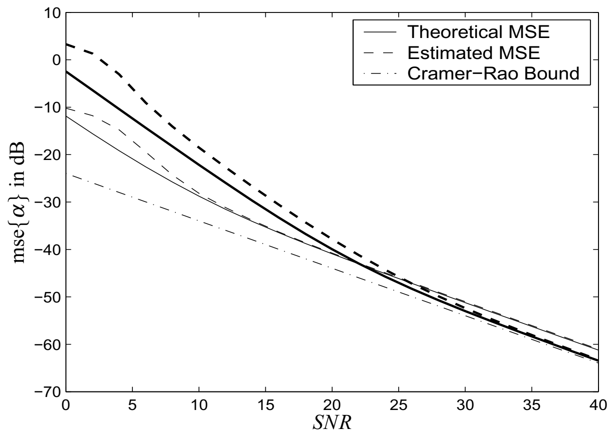

Figure 4: Theoretical and estimated mean square error of the damping factor as a function of the SNR. (—) KT method; (**—**) MP method.

5. CONCLUSION

We have presented a statistical analysis of two estimation methods: the Kumeresan-Tufts (KT) method and the Matrix Pencil (MP) approach. Our development is focused on the bias and variance of the damping factor. In particular, we have shown that the bias is generally not negligible. The presented analysis confirms the superiority of the MP method over the KT method in terms of variance at high SNR. However, the bias of the damping factor is better with the KT method. This feature is of great importance when it is necessary to apply the unit circle criterion, especially for multi–component signals.

REFERENCES

[1] R. Kumaresan and D. W. Tufts, "Estimating the parameters of exponentially damped sinusoids and pole-zero modeling in noise," *IEEE Trans. Acoust. Speech Signal Process.*, vol. 30, no. 6, pp. 833–840, 1982.

[2] Y. Hua and T. K. Sarkar, "Matrix pencil method for estimating parameters of exponentially damped/undamped sinusoids in noise," *IEEE Trans. Acoust. Speech Signal Process.*, vol. 38, no. 5, pp. 814–824, 1990.

[3] A. C. Kot, D. W. Tufts, and R. J. Vaccaro, "Analysis of linear prediction by matrix approximation," *IEEE Trans. Signal Process.*, vol. 41, no. 11, pp. 3174–3177, 1993.

[4] B. Porat and B. Friedlander, "On the accuracy of the Kumaresan-Tufts method for estimating complex damped exponentials," *IEEE Trans. Acoust. Speech Signal Process.*, vol. 35, no. 2, pp. 231–235, 1987.

[5] A. Okhovat and J. R. Cruz, "Statistical analysis of the Tufts-Kumaresan and principal Hankel components methods for estimating damping factors of single complex exponentials," in *Proc. Int. Conf. Acoust. Speech Signal Process.*, 1989, pp. 2286–2289.

[6] J. H. Wilkinson, *The algebraic eigenvalue problem*, Oxford University Press, 1965.

[7] E.-H. Djermoune and M. Tomczak, "Analysis of linear prediction-based methods in estimating damped sinusoids," Technical report, Centre de Recherche en Automatique de Nancy, UHP Nancy 1, France, 2003.

[8] A. V. Oppenheim and R. W. Schafer, *Digital signal processing*, Prentice Hall, Englewood Cliffs, NJ, 1975.