A NOVEL FREQUENCY ESTIMATOR AND ITS COMPARATIVE PERFORMANCES FOR SHORT RECORD LENGTHS

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ABSTRACT

Based on the linear prediction property of sinusoidal signals, a closed form unbiased frequency estimator for a real sinusoid in white noise is devised. The frequency estimate is determined from minimization of a modified least squares cost function. Computer simulations are included to evaluate the performance of the proposed estimator by comparing with existing closed form frequency estimators and the Cramér-Rao lower bound, particularly for short observation intervals.

1 INTRODUCTION

Frequency estimation of a real sinusoid in noise is an important problem that arises in a wide variety of applications such as angle of arrival estimation, demodulation of frequency-shift keying (FSK) signals, speech analysis and Doppler rate estimation [1]-[3]. The noisy discrete-time measurements of the sinusoid are represented as

$$x_n = \alpha \cos(\omega n + \phi) + q_n, \quad n = 0, 1, \ldots, N-1 \qquad (1)$$

where the noise q_n is assumed to be a zero-mean white process while α, $\omega \in (0, \pi)$ and $\phi \in [0, 2\pi)$ are unknown constants which denote the tone amplitude, radian frequency and phase, respectively. Without loss of generality, the sampling period is assigned to be unity. The task is to find ω from the N measurements of x_n, particularly when N is small, say of the order of 20 or less.

For a noisy complex sinusoid, it is well known that [1] the maximum likelihood (ML) estimate of frequency is obtained from the periodogram maximum. Kenefic and Nuttall [4] had extended the problem to ML frequency estimation of a real tone and the optimum estimator maximizes a highly nonlinear and multimodal cost function. For both cases, ML methods involve extensive computations and this will be prohibited in applications where rapid frequency estimation is required.

In this paper, we devise a computationally simple and accurate single real tone frequency estimator based on linear prediction of sinusoidal signals. Starting from the property that a noise-free sinusoid is a linear combination of its past two sampled values, a cost function whose minimum exactly corresponds to the sinusoidal frequency is developed in Section 2. A modified least squares estimator which gives unbiased as well as closed form frequency estimate is then derived. Several closed form instantaneous frequency estimators, namely, the modified covariance method [2],[5], Prony based techniques [6] and discrete energy separation algorithms (DESAs) [7] as well as the Cramér-Rao lower bound (CRLB) for frequency estimation are reviewed in Section 3. Simulation results are included in Section 4 to evaluate the short-time frequency estimation performance of the proposed algorithm by contrasting with the existing methods and CRLB. Finally, conclusions are drawn in Section 5.

2 PROPOSED FREQUENCY ESTIMATOR

A second order autoregressive (AR) model for representing x_n is

$$x_n = a_1 x_{n-1} + a_2 x_{n-2} \qquad (2)$$

where a_1 and a_2 are the AR coefficients. In the absence of noise, expanding the RHS of (1) in terms of $\cos(\omega n + \phi)$ and $\sin(\omega n + \phi)$ and equating the results with the LHS gives

$$\begin{aligned} a_1 \cos(\omega) + a_2 \cos(2\omega) &= 1 \\ a_1 \sin(\omega) + a_2 \sin(2\omega) &= 0 \end{aligned} \qquad (3)$$

It is easy to show that (3) has a unique solution of $a_1 = 2\cos(\omega)$ and $a_2 = -1$. This implies that in a noise-free condition, the frequency can be estimated accurately by searching the minimum of the mean square value of an error function e_n of the form

$$e_n = x_n - 2\cos(\hat{\omega}) x_{n-1} + x_{n-2} \qquad (4)$$

where $\hat{\omega}$ represents an estimate of ω. In general cases where $q_n \neq 0$, $E\{e_n^2\}$ can be calculated as

$$\begin{aligned} E\{e_n^2\} &= 4(\cos(\hat{\omega}) - \cos(\omega))^2 \sigma_s^2 \\ &\quad + 2(2 + \cos(2\hat{\omega}))\sigma_q^2 \end{aligned} \qquad (5)$$

where $\sigma_s{}^2 = \alpha^2/2$ denotes the tone power while σ_q^2 is the noise variance. Apparently, minimizing $E\{e_n^2\}$ with respect to $\hat{\omega}$ will not give an unbiased frequency estimation because the noise component of (5) is a function of $\hat{\omega}$. To remove the effect of noise, we employ a modified mean square error function $E\{\zeta_n^2\}$ where ζ_n is expressed as

$$\zeta_n = \frac{e_n}{\sqrt{2(2+\cos 2\hat{\omega})}} \tag{6}$$

Using (5) and (6), it can be shown that the performance surface $E\{\zeta_n^2\}$ has the minimum value of σ_q^2 at $\hat{\omega} = \omega$. From the received x_n, we form a least squares cost function $J_N(\hat{\omega})$:

$$J_N(\hat{\omega}) = \sum_{n=2}^{N-1} \zeta_n^2 \tag{7}$$

Differentiating $J_N(\hat{\omega})$ with respect to $\hat{\omega}$ and then setting the resultant expression to zero yields

$$\begin{aligned} \sum_{n=2}^{N-1} e(n)((x_n + x_{n-2})\cos(\hat{\omega}) + x_{n-1}) &= 0 \\ \Rightarrow 2A_N \cos^2(\hat{\omega}) - B_N \cos(\hat{\omega}) - A_N &= 0 \end{aligned} \tag{8}$$

where

$$A_N = \sum_{n=2}^{N-1} (x_n + x_{n-2}) x_{n-1}$$

and

$$B_N = x_{N-1}^2 - x_{N-2}^2 - x_1^2 + x_0^2 + 2\sum_{n=2}^{N-1} x_n x_{n-2}$$

The least squares frequency estimate is computed from the larger root of (7) as follows:

$$\hat{\omega} = \cos^{-1}\left(\frac{B_N + \sqrt{B_N^2 + 8A_N^2}}{4A_N}\right) \tag{9}$$

Substituting the expected values of A_N and B_N, that is, $(N-2)\alpha^2\cos(\omega)$ and $(N-2)\alpha^2\cos(2\omega)$, respectively, into (9), we get $\hat{\omega} = \omega$ which means that $\hat{\omega}$ is an unbiased estimate of ω.

3 EXISTING METHODS & BOUND

Existing closed form frequency estimators include the modified covariance method [2],[5], Prony based estimators [6] and discrete energy separation algorithms (DESAs) [7]. Basically, the first two schemes are linear predictive techniques as well. The modified covariance method minimizes the average of the estimated forward and backward prediction error powers while the Prony based estimators are derived from the application of Prony method [8], which is based on the observation that deterministic processes are perfectly predictable not only from an infinite number of past values but from a finite number of past values. On the other hand, the DESAs target on detecting modulations in amplitude-modulation-frequency-modulation (AM-FM) signals by estimating the product of their time-varying amplitude and frequency. The formulas for frequency estimation using these methods as well as the proposed estimator are shown in Table 1. There are two and three variants of the Prony based methods and DESAs, respectively, but they can be applied for $N = 4$ or $N = 5$ only. While general formulas are available for the proposed and modified covariance methods.

Table 1: Closed form frequency estimators

Algorithm	$\hat{\omega}$
DESA-1a (4pt) :	$\cos^{-1}\left(\frac{(x_2^2-x_1x_3)-(x_1^2-x_0x_2)+(x_1x_2-x_0x_3)}{2(x_2^2-x_1x_3)}\right)$
Prony (4pt):	$\cos^{-1}\left(\frac{x_1x_2-x_0x_3}{(x_1^2-x_0x_2)+(x_1x_3)}\right)$
DESA-1 (5pt):	$\cos^{-1}\left(\frac{2(x_2^2-x_1x_3)-(x_1^2-x_0x_2)-(x_3^2-x_2x_4)+x_1x_2-x_0x_3+x_2x_3-x_1x_4}{4(x_2^2-x_1x_3)}\right)$
DESA-2 (5pt):	$\frac{1}{2}\cos^{-1}\left(\frac{(x_2^2-x_0x_4)-(x_1^2-x_0x_2)-(x_3^2-x_2x_4)}{2(x_2^2-x_1x_3)}\right)$
Modified Prony (5pt):	$\cos^{-1}\left(\frac{(x_1x_2-x_0x_3)+(x_2x_3-x_1x_4)}{4(x_2^2-x_1x_3)}\right)$
Proposed Estimator (N pt):	$\cos^{-1}\left(\frac{B_N+\sqrt{B_N^2+8A_N^2}}{4A_N}\right)$ where $\begin{cases} A_N = \sum_{n=2}^{N-1}(x_n + x_{n-2})x_{n-1} \\ B_N = x_{N-1}^2 - x_{N-2}^2 - x_1^2 + x_0^2 + 2\sum_{n=2}^{N-1} x_n x_{n-2} \end{cases}$
Modified Covariance (N pt):	$\cos^{-1}\left(\frac{\sum_{n=1}^{N-1} x_n(x_{n-1}+x_{n+1})}{2\sum_{n=1}^{N-1} x_n^2}\right)$

Assuming that ω is not near 0 or π, the CRLB for single real tone frequency estimation is given by [3]

$$\text{CRLB} \approx \frac{12}{\text{SNR}\, N(N^2-1)} \tag{10}$$

where the bound decreases with $1/N^3$ asymptotically.

4 SIMULATION RESULTS

Computer simulations had been carried out to evaluate the frequency estimation performance for a single real tone of the proposed algorithm. We compared its mean square frequency errors (MSFEs) with the methods shown in Table 1 as well as CRLB. All simulation results provided were averages of 400 independent runs.

Figure 1 shows the 4-point frequency variances of the proposed, DESA-1a, modified covariance and Prony methods versus ω at a signal-to-noise ratio (SNR) of 20 dB. In this test, the source signal was a pure sinusoid with constant amplitude and phase where ϕ was uniformly distributed between $[0, 2\pi)$ at each trial, while the noise was a white Gaussian process. It can be seen that the proposed and modified covariance methods performed very similar and were superior to the other two algorithms for the whole range of ω. It is mainly because these two algorithms gave unbiased frequency estimation. Furthermore, they approached the CRLB when the frequency was between 0.4π and 0.6π. In Figure 2, the previous experiment was repeated for 5-point estimation and similar results were obtained. Note that the DESA-2 was not included because it cannot work for $\omega \in (0.5\pi, \pi)$ [7]. Figure 3 plots the MSFEs for SNR $\in (0, 60)$dB at $\omega = 0.234\pi$ and $N = 5$. Again, we see that the proposed and modified covariance methods were comparable and their variances were close to the CRLB for a wide range of SNRs. In addition, the performance of DESA-1 was much better than the modified Prony method. Figures 5 and 6 compare the estimation performances of the proposed and modified covariance algorithms at $N = 20$. From Figure 5, it is observed that the proposed method outperformed the modified covariance algorithm for a wide range of frequency at SNR=20dB. While in Figure 6, we see that the former had a smaller threshold SNR than the latter, although both performed similarly at very high SNRs.

An AM-FM signal in which the AM and FM amounts varied from 5 to 50% was also selected for the performance comparison. The signal was of the form [7]:

$$\left[1 + \kappa \cos\left(\frac{\pi}{100}\right)\right] \cdot \cos\left[\frac{\pi}{5}n + 20\lambda \sin\left(\frac{\pi}{100}\right)\right]$$

where $(\kappa, \lambda) \in \{(0.05i, 0.05j) : i, j = 1, \ldots, 10\}$ and $n = 1, \ldots, 400$. For each of the 100 data sequences in the sample set, the mean absolute and root-mean-square (RMS) values of the frequency estimation errors were computed. Tables 2 and 3 show the results for the cases of noise-free condition and SNR=20dB, respectively. We observe that in Table 2, all methods performed very well although the modified Prony algorithm was the best one. On the other hand, the proposed and modified methods outperformed the remaining schemes in the presence of noise. In fact, the performances of all the methods were not satisfactory but significant improvement could be achieved if a larger data length was used in the two unbiased algorithms.

5 CONCLUSIONS

A linear prediction based method has been developed for estimating the frequency of a real sinusoid embedded in noise. It is simple to implement and provides a closed form solution. In evaluating the short-time estimation performance of the proposed method, it is found that its accuracy is comparable with conventional algorithms in noise-free environments. While in noisy conditions, it is generally superior and can approach the CRLB.

Acknowledgments

This work is supported by a research grant from City University of Hong Kong (Project No. 9030874).

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Table 2: Frequency errors for AM-FM signal at SNR=∞

Algorithm	Mean Abs(%)	RMS(%)
Proposed Estimator (4pt)	0.91508	1.22610
DESA-1a (4pt)	0.82618	0.97295
Modified Covariance (4pt)	0.91384	1.22450
Prony (4pt)	0.38329	0.48998
Proposed Estimator (5pt)	0.38052	0.53158
DESA-1 (5pt)	0.32810	0.38867
DESA-2 (5pt)	0.40109	0.47015
Modified Covariance (5pt)	0.38050	0.53120
Modified Prony (5pt)	0.10594	0.14444

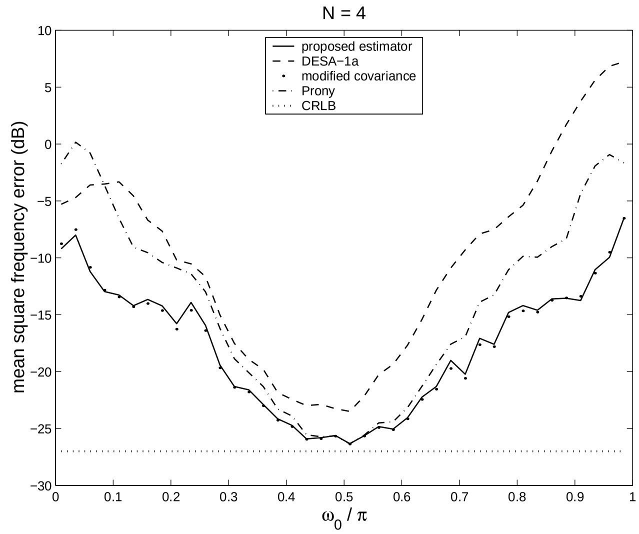

Figure 1: MSFEs versus ω_0 at SNR $= 20$dB

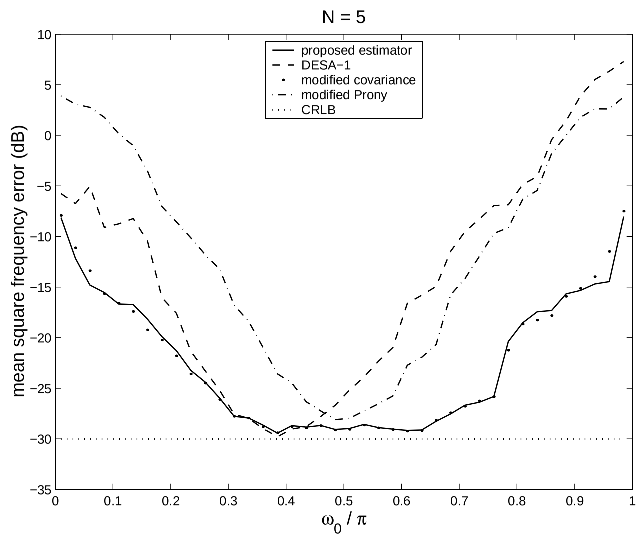

Figure 2: MSFEs versus ω_0 at SNR $= 20$dB

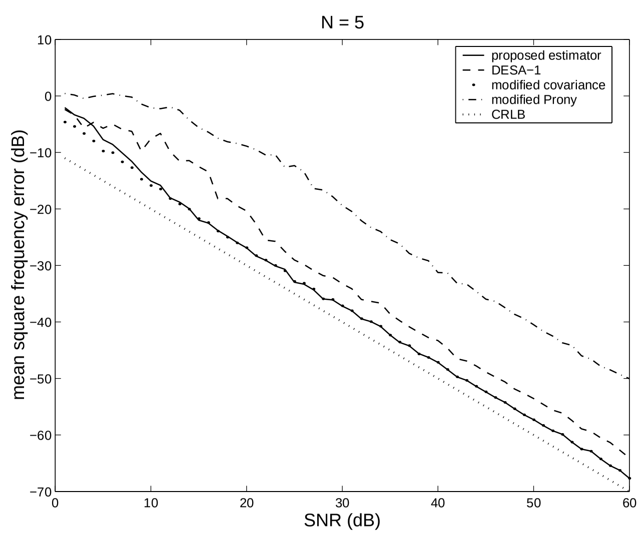

Figure 3: MSFEs versus SNR at $\omega_0 = 0.234\pi$

Table 3: Frequency errors for AM-FM signal at SNR=20dB

Algorithm	Mean Abs(%)	RMS(%)
Proposed Estimator (4pt)	25.777	37.582
DESA-1a (4pt)	52.201	79.310
Modified Covariance (4pt)	24.926	36.170
Prony (4pt)	45.601	62.893
Proposed Estimator (5pt)	15.603	23.877
DESA-1 (5pt)	25.318	43.922
DESA-2 (5pt)	32.746	47.400
Modified Covariance (5pt)	15.216	22.555
Modified Prony (5pt)	69.541	103.42
Proposed Estimator (10pt)	4.9243	6.9062
Modified Covariance (10pt)	6.9157	9.2871

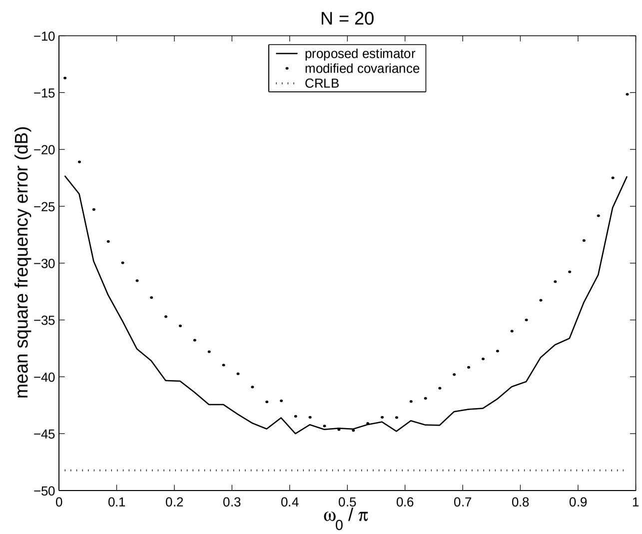

Figure 4: MSFEs versus ω_0 at SNR $= 20$dB

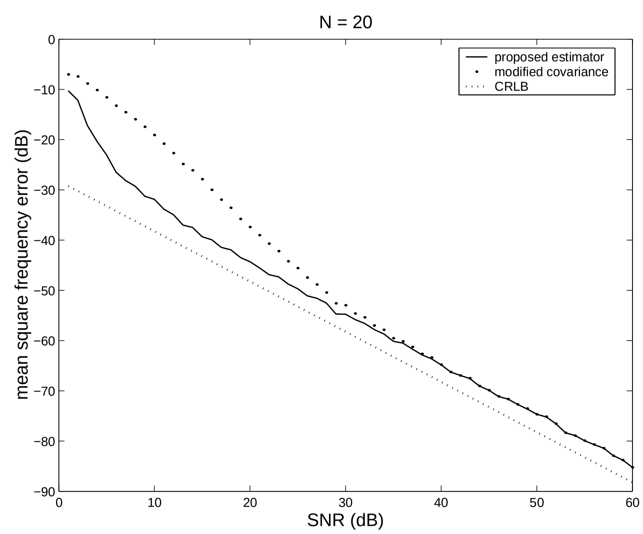

Figure 5: MSFEs versus SNR(dB) at $\omega_0 = 0.234\pi$