SNR ENHANCEMENT OF DAMPED EXPONENTIAL SIGNALS IN NOISE

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ABSTRACT

In this paper, it is shown that the use of a particular autocorrelation estimator, with fixed-length window, allows to improve the SNR of damped exponential signals in noise. A simple method based on a polynomial approximation of a geometric series is derived in order to compute the optimal window length in both single and multiple mode cases. Using multiple simulations, the results achieved with the original Kumaresan and Tufts method, which operates directly on data, are compared to those obtained when the same algorithm is applied to the autocorrelation estimates. It appears that, on signals consisting of one and two damped complex exponentials in white noise, the latter approach performs better than the Kumaresan-Tufts method when using the optimal window length.

1 INTRODUCTION

In the case of damped complex exponentials, the most popular parametric estimation method is the well-known Kumaresan and Tufts (KT) approach [1]. It performs a reduced rank pseudoinverse of the data matrix to get backward linear prediction parameters. By contrast, indirect approaches, that is using autocorrelation estimates like the Yule-Walker (YW) approach (see [2]), may be used although the latter is theoretically not appropriate in the case of damped signals. In this framework, Händel [3] has proposed some modifications to the original YW approach and showed, using Monte Carlo simulations, that his approach outperforms the KT method in terms of accuracy versus algorithmic complexity. Another way to deal with damped exponentials is to use an autocorrelation estimator that keeps constant the number of samples involved in the calculation of each lag [4]. The resulting estimator is closely related to the covariance method in linear prediction [5].

In this paper, our interest is focused on the latter autocorrelation estimator. It will be proved that in some cases, the window length may be chosen so as to improve the signal to noise ratio (SNR) of each component. A simple method to determine the optimal window length will be given, and multiple simulations will show the resulting improvements on the estimation of the signal parameters, and especially on the estimation variance, comparatively to the KT method.

The paper is organized as follows. In section 2 some theoretical results on the SNR improvement of damped complex exponentials in noise are presented. The simplest case of one damped exponential is discussed, and a polynomial method allowing to compute the optimal window length is proposed. The results are then extended to the multiple mode case. Multiple simulations using KT method both on data and autocorrelations are compared in section 3. Finally, the conclusions are given in section 4.

2 MAIN RESULTS

Consider the following complex signal composed of M damped exponentials

$$x(n) = \sum_{i=1}^{M} h_i z_i^n + e(n),\ n = 0, ..., N-1 \qquad (1)$$

where $z_i = \rho_i e^{j\omega_i}$ ($\rho_i < 1$) is the i^{th} component with complex amplitude h_i. $e(n)$ is a Gaussian complex white noise sequence with zero mean and variance σ_e^2. The initial SNR of each exponential component is defined by $SNR_i = |h_i|^2/\sigma_e^2$.

The autocorrelation estimator under study is defined (only for positive lags) by [4]

$$\hat{r}(k) = \frac{1}{L}\sum_{n=0}^{L-1} x^*(n)x(n+k),\ k = 0, ..., N-L \qquad (2)$$

Using the model defined in (1) and after straightforward calculations, one can show that $\hat{r}(k)$ is a combination of a deterministic term $\overline{r}(k)$ and a random term $\epsilon(k)$

$$\hat{r}(k) = \overline{r}(k) + \epsilon(k) = \sum_{i=1}^{M} h'_i z_i^k + \epsilon(k) \qquad (3)$$

where

$$\begin{aligned} h'_i &= \tfrac{h_i}{L}\textstyle\sum_{l=1}^{M} h_l^* f_L(z_i z_l^*) \\ \epsilon(k) &= \tfrac{1}{L}\textstyle\sum_{n=0}^{L-1}\Big[\sum_{l=1}^{M} h_l^* z_l^{*n} e(n+k) \\ &\quad + h_l z_l^{n+k} e^*(n) + e^*(n)e(n+k)\Big] \end{aligned} \qquad (4)$$

$f_L(z)$ is a geometrical series with ratio z:

$$f_L(z) = \sum_{n=0}^{L-1} z^n = \frac{1-z^L}{1-z} \tag{5}$$

Note that the autocorrelation model in equation (3) is the same that in equation (1) except that in the autocorrelation model, $\epsilon(k)$ is a correlated and non-stationary noise with mean and variance

$$\begin{aligned} E[\epsilon(k)] = &\ \sigma_e^2 \delta(k) \\ Var[\epsilon(k)] = &\ \frac{\sigma_e^2}{L^2} \sum_l \sum_r \Big[h_l^* h_r f_L(z_l^* z_r) \\ &+ h_l h_r^* (z_l z_r^*)^k f_L(z_l z_r^*) \Big] + \sigma_e^4 / L \end{aligned} \tag{6}$$

This equation shows that $Var[\epsilon(k)]$ depends on k. Let us consider the most pessimistic case, ie. $Var[\epsilon(k)]$ is constant and equal to $\max_k Var[\epsilon(k)]$ denoted by σ_ϵ^2. For simplicity, it will be supposed that σ_ϵ^2 is achieved when $k = 0$, which is not true in general, but will hold for the forthcoming simulations. As for the signal $x(n)$, one can define a new SNR for each component of the sequence $\hat{r}(k)$ as $SNR_i' = |h_i'|^2/\sigma_\epsilon^2$, and a SNR gain by the ratio

$$g_i(L) = SNR_i'/SNR_i \tag{7}$$

In the next sections, it will be shown that, in some cases, a SNR improvement ($g_i(L) > 1$) can be achieved. To illustrate this, let us consider the case of a single damped sinusoid in noise.

2.1 Single mode case

In the single mode case, $g(L)$ is given by

$$g(L) = \frac{SNR.f_L^2(|z|^2)}{L + 2SNR.f_L(|z|^2)} \tag{8}$$

Figure 1 shows the SNR gain versus the window length for some values of $|z|$. The use of the autocorrelation estimates as defined in (2) instead of the original data may improve the SNR, but may also reduce it. Generally speaking, when the SNR of the original signal is too low and the components are excessively damped, it is better to use the data. The SNR improvement is important when the original SNR is not too low and/or the complex exponential is not too much damped.

It may be observed from figure 1 that there exist an optimal window length L_{opt} that maximises $g(L)$. This length depends on both the initial SNR and the damping factor. The closer z is to the unit circle, the larger L_{opt} is, and conversely. The analytical computation of the optimal value of L is not a simple task due to the high nonlinear feature of both the numerator and the denominator of $g(L)$ in equation (8). Instead of using an iterative approach, one can consider an approximate solution using a polynomial approximation of the function $f_L(z)$ with respect to L. The Taylor series expansion of

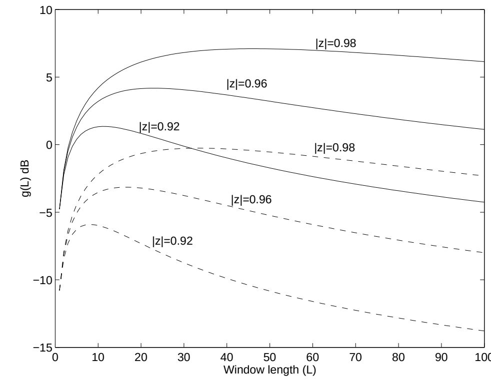


Figure 1: SNR gain versus window length for $SNR = 0$ dB (—) and $SNR = -10$ dB ($\cdots$).

the function $f_L(z)$ around $L = 0$ is

$$f_L(z) = \frac{1}{1-z} - \frac{1}{1-z} \sum_{j=1}^{\infty} \frac{(\log z)^j}{j!} L^j \tag{9}$$

A good approximation of $f_L(z)$ may be obtained by retaining only m terms from the infinite sum above leading to the finite order polynomial

$$\hat{f}_L(z) = \sum_{j=0}^{m} \alpha_j(z) L^j \tag{10}$$

where $\alpha_0(z) = \frac{1}{1-z}$ and $\alpha_j(z) = \frac{-(\log z)^j}{(1-z)j!}$, for $j = 1, ..., m$. For this approximation to hold in a large region of L, we must have $|\log z| < 1$. As will be seen in the next paragraph, when $|\log z| \geq 1$, the function $f_L(z)$ can be approximated by a fixed value or simply neglected. In the simple mode case, the argument z of the function $f_L(z)$ is to be replaced by $|z|^2$ which verify the condition $|\log |z|^2| < 1$ with the realistic assumption of a reasonable damping factor. By replacing $f_L(|z|^2)$ in (8) by its approximation $\hat{f}_L(|z|^2)$, it follows that

$$g(L) \approx \frac{SNR\Big(\sum_{j=0}^{m} \alpha_j(|z|^2)L^j\Big)^2}{L + 2SNR\sum_{j=0}^{m} \alpha_j(|z|^2)L^j} \tag{11}$$

Now, the problem of maximizing $g(L)$ with respect to L is reduced to the determination of a particular real zero of the first derivative of $g(L)$. This will be generalized in the next paragraph.

$\|z\|$	0.90	0.92	0.94	0.96	0.98	0.99	0.999
-10 dB	6	8	11	17	33	67	672
0 dB	9	11	15	23	46	93	927
10 dB	14	18	24	37	74	149	1494

Table 1: Optimal length as a function of $|z|$ and the SNR.

Table 1 shows the optimal window lengths obtained by the polynomial approximation with order $m = 10$. It may be noted that L_{opt} is an exponentially increasing function when $|z|$ tends to 1.

2.2 Multiple mode case

First, begin by rewriting the expressions of h'_i and σ_ϵ^2

$$\begin{array}{l} h'_i = \frac{h_i}{L}\sum_l h_l^* f_L(z_i z_l^*) \\ \sigma_\epsilon^2 = \frac{\sigma_e^4}{L} + \frac{2\sigma_e^2}{L^2} Re\Big\{ \sum_{l,r} h_l^* h_r f_L(z_l^* z_r) \Big\} \end{array} \quad (12)$$

In this case the argument of the function $f_L(z)$ is not always real. Indeed, the cross products $z_l^* z_r$ for $l \neq r$ are complex valued leading to an oscillatory behavior of the sums $g_L(z_l^* z_r)$ and consequently their polynomial approximations are very poor especially when $\arg(z_l^* z_r)$ is large (high frequency oscillations). When $|\log z_l^* z_r| < 1$, a polynomial approximation of the function performs well. If $|\log z_l^* z_r| \geq 1$, we have found that $f_L(z_l^* z_r)$ can be replaced by its steady state value, ie. $1/(1 - z_l^* z_r)$ or simply 0. This is due to the fact that the interaction energy between two remote poles is negligible comparatively to the energy of one or the other of the two poles. With this in mind, $f_L(z_i z_l^*)$ may be now replaced in the expression of h'_i by its approximation of order m

$$h'_i \approx \frac{h_i}{L}\sum_{j=0}^{m}\Big[\sum_l h_l^* \alpha_j(z_i z_l^*)\Big]L^j = \frac{h_i}{L}\sum_{j=0}^{m}\lambda_j(z_i)L^j \quad (13)$$

In the same manner, σ_ϵ^2 may be written as

$$\begin{array}{ll} \sigma_\epsilon^2 & \approx \frac{\sigma_e^4}{L} + \frac{2\sigma_e^2}{L^2}\sum_{j=0}^m Re\Big\{ \sum_{l,r} h_l^* h_r \alpha_j(z_l^* z_r)\Big\} L^j \\ & \approx \frac{\sigma_e^4}{L} + \frac{2\sigma_e^2}{L^2}\sum_{j=0}^m \gamma_j L^j \end{array} \quad (14)$$

The SNR gain becomes a ratio of two polynomials

$$g_i(L) \approx \frac{\left|\sum_{j=0}^m \lambda_j(z_i)L^j\right|^2}{L + 2\sum_{j=0}^m \gamma_j L^j} = \frac{\sum_{j=0}^{2m}\mu_j(z_i)L^j}{\sum_{j=0}^m \nu_j L^j} \quad (15)$$

The optimal value of L that corresponds to the maximum of the function $g_i(L)$ $(dg_i(L)/dL = 0)$ is a real and positive root of the following polynomial of order $3m-1$

$$\begin{array}{ll} P_i(L) & = \sum_{j=1}^{2m} j\mu_j(z_i)L^{j-1}.\sum_{j=0}^m \nu_j L^j \\ & - \sum_{j=0}^{2m}\mu_j(z_i)L^j.\sum_{j=1}^m j\nu_j L^{j-1} \end{array} \quad (16)$$

As an example, figure 2 shows the SNR gains for two damped exponentials with same SNR (0 dB) and different damping factors ($z_1 = 0.96e^{j2\pi 0.08}$ and $z_2 = 0.98e^{-j2\pi 0.08}$). So $|\log z_1^* z_2| = 1.0072$, which is greater than 1. The approximation curve obtained using a polynomial of order $m = 30$ is a smoothed version of the gain when no approximation is used. Using the polynomial approximation, optimal values of L with respect to the two exponentials are 23 and 54 respectively, which approximately coincides with the theoretical maxima of SNR gains. Here the value of L that ensures a SNR gain superior to 0 dB for both components can be set to L_{1opt}. This choice is not optimal for the second component but a larger value of L could reduce the SNR of the first one. Note that, compared to the single mode case, the optimal window length for the second component has moved, and is the same for the first one.

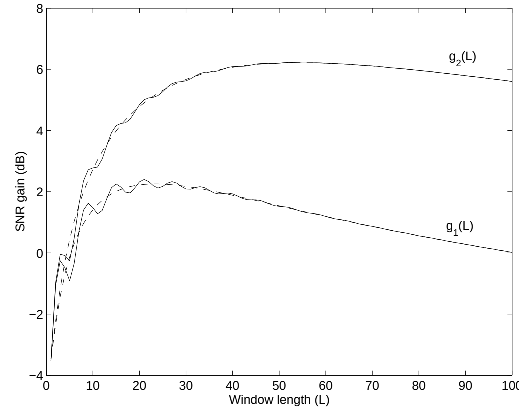


Figure 2: SNR gains on signal containing two damped exponentials at 0 dB. (—) True gain using exact expression of $f_L(z)$. $(\cdots)$ Polynomial approximation, $m = 30$.

3 SIMULATIONS

In this section, some results obtained on the pole estimation of damped exponential signals from noisy data are presented. Comparisons will be made between KT method applied on data (KT-data) and the same method operating on autocorrelation estimates defined in equation (2) (KT-AC). The same prediction order is used in both cases.

The first simulation consists of one damped complex exponential located in polar coordinates at $(0.98, 2\pi 0.2)$ with amplitude 1. The white noise variance is set in order to have a SNR of 0 dB. The prediction order for both methods is set to $p = 10$. Table 2 shows the results achieved using the two methods with 500 noise realizations. The estimated variance of the estimated pole location achieved when using KT-AC ($L = L_{opt} = 46$) is better than that obtained with KT-data, when the parameter c is the same for the two methods (c denotes the number of rows of data and autocorrelation matrices). In fact, KT-AC method uses L more signal samples than KT-data. But, even if c_{data} is chosen equal to $c_{\text{autoc}} + L$, it can be seen the results are still better with KT-AC. Due to the non-whiteness of the noise $\epsilon(k)$, one can expect a large bias when using the autocorrelations. But the results achieved show that the bias (and thus the MSE) is also better compared to that obtained on data.

	KT-data ($\times 10^{-3}$)			KT-AC ($\times 10^{-3}$)		
c	Var	$\lvert Bias\rvert^2$	MSE	Var	$\lvert Bias\rvert^2$	MSE
10	8.66	1.90	10.56	1.06	1.93	2.99
15	4.48	1.59	6.07	0.69	0.85	1.54
20	3.21	1.56	4.78	0.40	0.54	0.94
25	5.67	1.97	7.64	0.47	0.38	0.85
30	2.24	2.04	4.28	0.52	0.25	0.76
60	1.52	3.76	5.29	0.22	0.05	0.26
70	1.43	4.61	6.04	0.21	0.01	0.22

Table 2: Results achieved on signal containing one damped complex exponential.

The second simulation signal contains two damped sinusoids at $(0.96,\ 2\pi 0.08)$ and $(0.98,\ -2\pi 0.08)$ in polar coordinates, with unit amplitudes. The SNR is set to 0 dB. Table 3 shows the results obtained with the KT-AC method ($p = 10$ and $c = 20$) using several window lengths. The theoretical results are confirmed since the optimal (with regard to the variance) window lengths for the two components are $L \approx 23$ and $L \approx 54$, respectively.

L	$Varz_1$	$Varz_2$	$MSEz_1$	$MSEz_2$
15	-21.69	-31.58	-21.41	-31.03
20	-21.34	-32.04	-21.04	-30.87
23	-21.74	-32.15	-21.23	-30.93
30	-19.28	-30.77	-18.95	-29.49
40	-19.84	-31.49	-19.24	-29.39
50	-19.32	-33.70	-18.54	-29.83
54	-20.49	-33.62	-19.65	-29.58
60	-18.76	-31.32	-18.12	-28.50

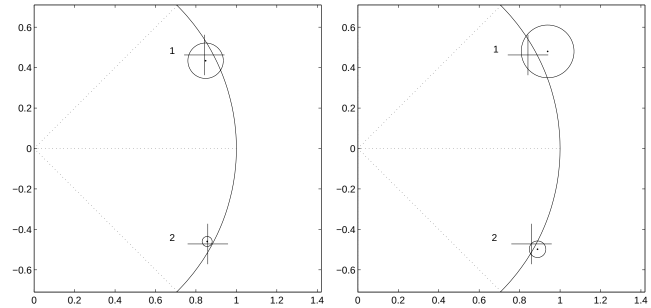
Table 3: Variance and MSE of estimated poles using KT-AC (in dB).

Comparison with both the variances and MSEs achieved with KT-data in table 4 shows that the autocorrelation method yields a reduction of the variance of the second component of about 10 dB. The KT-data method does not reach the performances of the KT-AC method with the same prediction order and signal length. This suggests that to attain the same performances, the prediction order of the KT-data method must be raised, thus increasing the computational burden of the estimation procedure.

c	$Varz_1$	$Varz_2$	$MSEz_1$	$MSEz_2$
10	-17.11	-20.64	-15.86	-19.76
15	-17.45	-23.41	-16.10	-22.07
20	-18.01	-23.58	-16.31	-21.92
25	-18.65	-26.35	-16.49	-23.76
30	-16.85	-25.45	-14.70	-22.75
60	-15.23	-26.39	-12.48	-21.82
70	-14.89	-26.69	-11.82	-21.44

Table 4: Variance and MSE of estimated poles using KT-data (in dB).

Figure 3 shows the positions of the poles on the z-plane. The circles are centered on the mean of estimated pole locations, their radii being equal to the estimated standard deviations. The crosses figure the true pole locations. Another consequence of the noise reduction observed with KT-AC method lies in the position of the poles relatively to the unit circle. In backward prediction, the KT-data estimates fall inside the unit circle (and outside after reflection, as shown on figure 3(b)), and thus cannot be separated from the extraneous zeros introduced by the use of a prediction order larger than the number of components. That is not the case with KT-AC method for the signal under study.



(a) autocorrelations (b) data

Figure 3: Positions and variances of the estimated poles. (a) $L = 23$, $p = 10$ and $c = 20$, (b) $p = 10$ and $c = 20$.

4 CONCLUSION

A particular autocorrelation estimator has been studied in the case of noisy damped exponential signals. It is shown that under the assumption of reasonable noise level and damping factors, the use of this estimator leads to an increase of the SNR. A polynomial approximation method has been derived to compute the optimal window length associated to the autocorrelation estimator. This approximation can be used after an initial estimation since it requires the knowledge of signal parameters. Then, it is shown using multiple simulations that the SNR enhancement results in a more accurate and reliable parameter estimation in the case of KT method. Moreover, the unit circle criterion, allowing to separate the signal and noise poles, operates in better conditions.

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