CROSS-TERMS FREE FORMS OF SOME QUADRATIC AND HIGHER ORDER TIME-FREQUENCY REPRESENTATIONS

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ABSTRACT

The S-method based time-frequency analysis is presented. In the case of multicomponent signals, it can produce a sum of the pseudo Wigner distributions of each signal component separately. The only condition is that the spectrogram is cross-terms free. The realization is based on the short-time Fourier transform, for which well studied software methods and hardware systems exist, what makes this method attractive for applications and implementations in time-frequency analysis, including higher-order distributions.

1 INTRODUCTION AND REVIEW

Short-time Fourier transform is the oldest and most widely used tool for time-frequency analysis, [1, 2, 3, 4, 5]. In order to overcome some drawbacks of this transform, various quadratic representations (distributions) have been used, [3, 5]. The most prominent member of these distributions is the Wigner distribution. It produces the best auto-term concentration, but its disadvantage lies in very exhibited cross-term effects, [3, 5, 6]. That was the reason why many reduced interference distributions (RID) have been defined, [7]; the S-method is one of the recently defined ones. It is derived from the relation between the short-time Fourier transform and the pseudo Wigner distribution [8], and described in detail in [9, 10, 11, 12, 14, 16, 17, 22]. Multidimensional generalization of the S-method is presented in [13]. The S-method may be efficiently used in implementations of the time-scale distributions [9, 18], polynomial Wigner-Ville distributions [20, 17] and local polynomial distributions [21]. The fact that it uses, as a basic step, the short-time Fourier transform, for which the well studied software methods and hardware systems exist, makes the S-method attractive for applications and implementations in time-frequency analysis. In contrast to the other reduced interference distributions which are usually derived from the condition that the marginal properties are preserved (what inherently leads to the auto-term degradation, with respect to the Wigner distribution, [6, 12]), the S-method is derived with the goal to preserve the same auto-terms as in the Wigner distribution, avoiding cross-terms. After a short review of the basic relations, we will provide some new theoretical considerations concerning the signal dependent form of the S-method, Sec.2, along with its generalizations to the higher order forms. This form may further significantly improve the S-method properties (auto-terms concentration, cross-terms elimination, noise influence...), without sacrificing the implementation simplicity, Sec.3.

A definition of the short-time Fourier transform is

$$STFT(t,\omega)=\int_{\tau} f(t+\tau)w(\tau)e^{-j\omega\tau}d\tau, \tag{1}$$

while the Wigner distribution, in its pseudo form, is given by:

$$WD(t,\omega)=\int_{\tau} f(t+\frac{\tau}{2})f^{*}(t-\frac{\tau}{2})w(\frac{\tau}{2})w(-\frac{\tau}{2})e^{-j\omega\tau}d\tau. \tag{2}$$

Relationship between (1) and (2), as derived in [8], is

$$WD(t,\omega)=\int_{\theta} STFT(t,\omega+\theta)STFT^{*}(t,\omega-\theta)\frac{d\theta}{\pi}. \tag{3}$$

This relation has led to the S-method, an efficient tool for time-frequency analysis, [8, 9, 10, 11, 12, 16, 17, 22]:

$$SM(t,\omega)=\int_{\theta} P(\theta)STFT(t,\omega+\theta)STFT^{*}(t,\omega-\theta)\frac{d\theta}{\pi}. \tag{4}$$

where $P(\theta)$ is a finite frequency domain window (we also assume rectangular), $P(\theta)=0$, for $|\theta|>L_P$. Kernel function in (θ,τ) domain of the S-method is given by $c(\theta,\tau)=P(\theta/2)*_{\theta}A_{ww}(\theta,\tau)/2\pi$, where $A_{ww}(\theta,\tau)$ is the ambiguity function of $w(\tau)$. It is generally a non-separable function, meaning that the S-method does not belong to the class of distributions commonly referred to as the smoothed pseudo Wigner distributions. A comparison of the S-method with other distributions from the Cohen class [3, 5], with respect to the auto-term representation, may be found in [12]. The S-method, given by (4), may produce the representation of a multicomponent signal such that the distribution of each component is its Wigner distribution, avoiding cross-terms.

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Here, we will introduce a signal dependent form of the S-method and provide its mathematical treatment.

2 THEORY

Theorem: *Consider signal* $f(t) = \sum_{i=1}^{M} f_i(t)$, *where* $f_i(t)$ *are monocomponent signals. Assume that the short-time Fourier transform of each component lies inside*[1] *the region* $D_i(t,\omega)$, $i = 1, 2, ..., M$. *Denote the length of* $i-th$ *region along* ω, *for a given* t, *by* $2W_i(t)$, *and its central frequency by* $\omega_{0i}(t)$. *The S-method of* $f(t)$ *produces a sum of the pseudo Wigner distributions* $WD_i(t,\omega)$, $i = 1, 2, ..., M$, *of each signal's component separately:*

$$SM(t,\omega) = \sum_{i=1}^{M} WD_i(t,\omega), \tag{5}$$

if the regions $D_i(t,\omega)$, $i = 1, 2, ..., M$, *do not overlap* ($D_i(t,\omega) \cap D_j(t,\omega) = \emptyset$ *for* $i \neq j$) *and the width of rectangular window* $P(\theta)$, *for a point* (t,ω), *is defined by:*

$$L_P(t,\omega) = W_i(t) - |\omega - \omega_{0i}(t)| \tag{6}$$

for $(t,\omega) \in D_i(t,\omega)$, $i = 1, 2, ..., M$ *and* 0 *elsewhere.*

Signal dependent form of the S-method reads:

$$SM(t,\omega) = \int_{\theta} P_{(t,\omega)}(\theta) STFT(t, \omega+\theta) STFT^*(t, \omega-\theta) \frac{d\theta}{\pi}. \tag{7}$$

Proof: Consider a point (t,ω), inside a region $D_i(t,\omega)$ such that $D_i(t,\omega) \cap D_j(t,\omega) = \emptyset$ for $i \neq j$ (meaning cross-terms free spectrogram). The integration interval in (3), for the $i-th$ signal component is symmetrical with respect to $\theta = 0$. It is defined by the smallest absolute value of θ for which $\omega + \theta$ or $\omega - \theta$ falls outside $D_i(t,\omega)$, i.e., $|\omega + \theta - \omega_{0i}(t)| \geq W_i(t)$ or $|\omega - \theta - \omega_{0i}(t)| \geq W_i(t)$. For $\omega > \omega_{0i}(t)$ and positive θ, the integration limit is reached first in $|\omega + \theta - \omega_{0i}(t)| \geq W_i(t)$ for $\theta = W_i(t) - (\omega - \omega_{0i}(t))$. For $\omega < \omega_{0i}(t)$ and positive θ, the limit is reached first in $|\omega - \theta - \omega_{0i}(t)| \geq W_i(t)$ for $\theta = W_i(t) - (\omega_{0i}(t) - \omega)$. Thus, having in mind the interval symmetry, integration limit which produces value of integral (7) the same as value of (3), over region $D_i(t,\omega)$, is given by (6). Therefore, for $(t,\omega) \in D_i(t,\omega)$ we have $SM(t,\omega) = WD_i(t,\omega)$. Since $L_P(t,\omega) = 0$ for $(t,\omega) \notin D_i(t,\omega)$, $i = 1, 2, ..., M$, relation (5) follows.

Note 1: Real M-component signals may be considered as $2M$-component complex signals with each region $D_i(t,\omega)$ being associated with $D_{i+M}(t,-\omega)$.

Note 2: The width defined by (6) is the minimal width which can achieve concentration as in the Wigner distribution, meaning that it is optimal with respect to the noise influence, cross-terms elimination, and the number of operations in numerical realizations.

Corollary 1: *Any window* $P(\theta)$ *with* **constant width** $L_P \geq \max_{\omega,t} \{L_P(\omega,t)\}$ *produces* $SM(t,\omega) = \sum_{i=1}^{M} WD_i(t,\omega)$, *if the regions* $D_i(t,\omega)$, $i = 1, 2, .., M$, *are at least* $2L_P$ *apart along the frequency axis, i.e.,* $|\omega_{0i}(t) - \omega_{0j}(t)| > W_i(t) + W_j(t) + 2L_P$, *for each* i, j *and* t, (S-method with constant window width, [9, 10, 11, 13]).

[1] *"Inside" means that the borders of* $D_i(t,\omega)$ *are not included.*

Corollary 2: *The S-method of signal* $f(t)$, *defined in the Theorem, lies inside the regions* $D_i^{(s)}(t,\omega)$ *for which* $D_i^{(s)}(t,\omega) \subseteq D_i(t,\omega)$, $i = 1, 2, ..., M$ *holds.*

3 HIGHER ORDER FORMS

In order to improve some properties of the signal's time-frequency representation, various higher order distributions have been introduced. From the practical point of view, of special interest are those which may be reduced to the two dimensional time-frequency plane (either by slicing or projections, [19]). One of these distributions is the L-Wigner distribution, [9, 10, 11, 19, 22]. The $L-th$ order L-Wigner distribution, in its pseudo form, is given by:

$$LWD_L(t,\omega) = \int_{\tau} w_L(\tau) f^L\left(t + \frac{\tau}{2L}\right) f^{L*}\left(t - \frac{\tau}{2L}\right) e^{-j\omega\tau} d\tau. \tag{8}$$

It may be expressed in terms of its $L/2-th$ order, as [9, 10, 11, 22]:

$$LWD_L(t,\omega) = \int_{\theta} LWD_{L/2}(t, \omega+\theta) LWD_{L/2}(t, \omega-\theta) \frac{d\theta}{\pi}. \tag{9}$$

Corollary 3: *The L-Wigner distribution of signal* $f(t)$, *defined in the Theorem, realized recursively according to*

$$LWD_L(t,\omega) = \int_{\theta} P_{(t,\omega)}(\theta) LWD_{L/2}(t, \omega+\theta) LWD_{L/2}(t, \omega-\theta) \frac{d\theta}{\pi}, \tag{10}$$

with the rectangular window $P_{(t,\omega)}(\theta)$ *whose width is defined by (6), is equal to the sum of the L-Wigner distributions of each component separately*

$$LWD_L(t,\omega) = \sum_{i=1}^{M} LWD_{L,i}(t,\omega), \tag{11}$$

where $LWD_{L,i}(t,\omega)$ *is the L-Wigner distribution of signal's* $i-th$ *component.*

Proof: Based on Corollary 2, the S-method, as well as any other L-Wigner distribution with $L > 1$, realized according to (10), lies inside the regions $D_i^{(L)}(t,\omega) \subseteq D_i(t,\omega)$, $i = 1, 2, ..., M$. If $D_i(t,\omega)$ do not overlap then $D_i^{(L)}(t,\omega)$ also do not overlap and (11) easily follows from the Theorem.

Note 3: Once we have obtained cross-term free L-Wigner distribution of the second order and cross-terms free pseudo Wigner distribution, we may easily realize the cross-terms free fourth order polynomial Wigner-Ville distribution, [20, 19, 17].

4 ALIASING EFFECTS

Corollary 4: *Consider sampled signal $f(t)$ in its analog notation, $f_s(t) = \sum_{n=-\infty}^{\infty} Tf(nT)\delta(t-nT)$. Assume that the short-time Fourier transform of signal $f(t)$ lies inside region $D(t,\omega)$. Since the short-time Fourier transform of $f_s(t)$ lies inside $D_s(t,\omega) = \bigcup_{k=-\infty}^{\infty} D(t,\omega+2k\pi/T)$, it may be formally considered as a multicomponent signal. Therefore, according to the Theorem, the S-method is equal to the signal's pseudo Wigner distribution, inside the basic period $\pi/T < \omega \leq \pi/T$:*

$$SM(nT,\omega) = WD(nT,\omega),$$

under the condition $D(t,\omega+2i\pi/T) \cap D(t,\omega+2j\pi/T) = \emptyset$ for $i \neq j$, (i.e., **the short-time Fourier transform is alias free***), and the width of $P_{(t,\omega)}(\theta)$ is defined as in (6).*

Note 4: This holds in the case of a sampled multicomponent signal, as well. The proof is evident with $D(t,\omega) = \bigcup_{i=1}^{M} D_i(t,\omega)$.

5 PRACTICAL HINTS

Numerical realization of the S-method (7) is very simple, according to:

$$SM(n,k) = SPEC(n,k) + 2\sum_{i=1}^{L_P(n,k)} Real[STFT(n,k+i)STFT^*(n,k-i)] \tag{12}$$

There are two possibilities to implement this summation:

1) With a signal independent $L_P(n,k) = L_P$. This way is very simple, and from our experience very efficient. Theoretically, in order to get the pseudo Wigner distribution for each component, the length L_P should be such that $2L_P$ is equal to the width of the widest auto term. This will guaranty cross-terms free distribution for all components which are at least $2L_P$ samples apart. For components and time instants where this condition is not satisfied, the cross-terms will appear, but still in a reduced form. In practice this means taking only a few samples, for example $L_P = 1, 2,$ or 3, what will significantly improve the time-frequency representation and will not introduce the cross-terms, except in a reduced form for very close components.

2) With a signal dependent $L_P(n,k)$ where the summation, for each point (n,k), lasts until zero value of $STFT(n,k+i)$ or $STFT(n,k-i)$ is detected (practically, that means a value of $STFT(n,k+i)$ or $STFT(n,k-i)$ smaller than a reference value R). If zero value may be expected within a single auto-terms, then the summation lasts until two subsequent zero values of $STFT(n,k+i)$ or $STFT(n,k-i)$ are detected. Here, we will propose two ways for determining the reference value R. One is based on the a priori knowledge about the signal and its STFT range. This is especially applicable in the cases when the signal is obtained as output of an A/D converter, and used in the fixed-point hardware implementations of time-frequency algorithms. In this case the signal must be within a priori prescribed range in order to optimally use the available converter and hardware registers. Here, the determination of reference level is possible on the a priori basis, as a few percents of the maximal expected spectrogram's value. If the a priori knowledge about the signal's spectrogram range is not reliable, then the reference level may be defined as a few percents of the spectrogram's maximal value at a considered instant n, $R_n = \max_k \{SPEC(n,k)\}/Q^2$. Both of these approaches may be easily implemented, even in real time or in the VLSI forms. For numerical examples and specific implementations see [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 23]. Here we presented the S-method based time-frequency analysis of a three-component signal, Figure 1. The higher order distributions are realized using components amplitude normalization [15, 23]. In hardware implementations, as well as in the cases when signal components partially overlap, we found as very convenient the application of the frequency domain window with predefined maximal allowable width, i.e., if $L_P(n,k) \geq L_{P\max}$, then $L_P(n,k) = L_{P\max}$. This does not significantly degrade the auto-terms concentration, and at the same time helps in the realization, and in the cross-terms reduction (when two components partially overlap).

6 CONCLUSION

An analysis of the S-method, which may in the case of multicomponent signals produce a sum of the Wigner distributions of each component separately, is presented. The theory is extended to the cross-terms free higher order distribution realizations.

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Figure 1: Time-frequency representation of a multicomponent signal: a) Wigner distribution, b) The S-method based representation, $L = 1$, c) Cross-terms free higher order distribution with $L = 2$, d) Cross-terms free higher order distribution with $L = 8$.

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