QUANTIZATION EFFECTS IN IMPLEMENTATION OF DISTRIBUTIONS FROM THE COHEN CLASS

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ABSTRACT

The paper presents an analysis of the finite register length influence on the accuracy of results obtained by the time-frequency distributions (TFDs). In order to measure quality of the obtained results, the variance of the proposed model is found, signal-to-quantization noise ratio (SNR) is defined and appropriate expressions are derived. Floating- and fixed-point arithmetic are considered. It is shown that commonly used reduced interference distributions (RID) exhibit similar performance with respect to the SNR. We have also derived the expressions establishing relationship between the number of bits and required quality of representation (defined by the SNR), which may be used for register length design in hardware implementation of TFDs.

1 INTRODUCTION AND REVIEW

Realizations of the TFDs admit both hardware and software implementation. For real time applications it is often necessary to use hardware implementation which gives rise to some new issues, one of the most important being the selection of appropriate register length. Shorter register length requires less hardware, but it may produce lower resolution and range. Registers of finite length, used to represent signals in TF analysis, also introduce quantization errors, [9], which rapidly accumulate and may adversely affect the obtained results. Rounding of arithmetic operations results also introduce errors, whose influence to the final result depends on the chosen number representation (fixed- or floating-point). Fixed-point arithmetic is characterized by a lower range and is more sensitive to addition overflow, [5, 9]. To overcome this problem, the floating-point representation and arithmetic are used. It significantly extends dynamic range, but for a given register length it must be done at the expense of the precision; thus a trade-off between the lengths of mantissa and exponent in the hardware implementation should be carefully considered.

This paper extends the analysis of finite register length effects of [5, 12] to the TFDs from the general Cohen class of representations (CD), [4, 13], for both floating- and fixed-point representations. TFDs' variance and the SNR have been derived and used as criteria for quantitative comparison of various TFDs from the CD, with regard to the finite word-length effects. Quasistationary random signals, as a special form of the nonstationary signals, have been analyzed. The underlying theme of this paper will be the conflict between the desire to obtain fine quantization (defined by the SNR) and wide dynamic range while holding the register length fixed. This discussion may be used in the optimization of register length which is an economic factor in hardware implementation of considered TFDs.

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2 VARIANCE OF THE CD'S ESTIMATOR

Discrete form of the CD of signal $f(n)$, is defined by, [6, 11, 13]:

$$\begin{gathered} C_f(n,k;\varphi)=\sum_{i=-L}^{L-1} r_f(n,i)e^{-j\frac{4\pi}{N}ki} \\ r_f(n,i)=\sum_{m=-L}^{L-1}\varphi(m,i)f(n+m+i)f^*(n+m-i) \end{gathered} \tag{1}$$

where $r_f(n,i)$ is the generalized autocorrelation function of $f(n)$, while $N=2L$ is the duration determined by the time-lag kernel $\varphi(m,i)$ width along the time and lag axis, [4, 13]. In order to analyze the influence of registers' lengths to the accuracy of results obtained by TFDs from the CD, it is necessary to find the variance of the Cohen's estimator.

Let us assume complex, nonstationary, stochastic process $f(n)$ with independent real and imaginary parts with equal variances $\sigma_{fn}^2/2$, $E\{f(n_1)f(n_2)\}=E\{f^*(n_1)f^*(n_2)\}=0$. In order to accomplish an adequate expression for the second moments of estimator (1), we should apply hypothesis of quasistationarity of analyzed nonstationary signal, [8], $E\{f(n+i_1)f^*(n+i_2)\}\cong\sigma_{fn}^2\delta(i_1-i_2)$. Reasons for that are in unavailability of ensemble averages of random signal and in impossibility of using ergodicity concept of the observed nonstationary processes.

Consequently, the general expression for covariance of the Cohen's estimator can be found as:

$$\begin{gathered} Cov\approx\sum_{i,m=-L}^{L-1}[\varphi(m,i)e^{-j\frac{4\pi}{N}k_1 i} **_{m,i}\varphi^*(-m,-i) \\ \times e^{-j\frac{4\pi}{N}k_2 i}]R_{n_a}(\tau+m+i)R_{n_b}^*(\tau+m-i) \end{gathered} \tag{2}$$

where $R_n(\tau)$ is the autocorrelation of the signal $f(n)$, $n_a=(n_1+n_2)/2+(i_1+i_2)/2$, $n_b=(n_1+n_2)/2-$

$(i_1+i_2)/2$ and $\tau = n_1 - n_2$, and $*_m$ and $*_i$ denote convolution over m and i, respectively. In the above equation sign '$\approx$' is used in order to apply hypothesis of quasistationarity. Supposing that τ is small enough (what is necessary in order to find variance), we can conclude, [8]: $R_{n_a}(\tau) \approx R_{n_b}(\tau) \approx R_{n_0}(\tau)$, where: $n_0 = (n_1+n_2)/2$. Assuming $f(n)$ is a white random process with variance σ_{fn}^2, we have:

$$Cov \approx \sigma_{fn_0}^4 \sum_{i,m=-L}^{L-1} \varphi(m,i)\varphi^*(m+\tau,i) \tag{3}$$

and for $n_1 = n_2 = n$ and $k_1 = k_2 = k$, variance of the estimator of Cohen's class in the final form:

$$\sigma^2(k) \approx \sigma_{fn}^4 E_\varphi \tag{4}$$

where $E_\varphi = \sum_{i,m=-L}^{L-1} |\varphi(m,i)|^2$ is the energy of kernel $\varphi(m,i)$. Now, we may found the variance of the CD of signal $x(n) = f(n) + \nu(n)$, where $\nu(n)$ denotes additive, complex, random, white noise with independent real and imaginary parts with equal variances $\sigma_\nu^2/2$. Supposing that the signal and noise processes are uncorrelated, the variance of CD's estimator may be directly obtained from (4), by replacing the signal variance σ_{fn}^2 with the sum of the signal and noise variances $(\sigma_{fn}^2 + \sigma_\nu^2)$:

$$\sigma_{xx}^2(k) \approx (\sigma_{fn}^2 + \sigma_\nu^2)^2 E_\varphi \tag{5}$$

It is very interesting to emphasize that the variance $\sigma_{xx}^2(\omega)$, in the case of uniformly distributed processes, is slightly different from the case of white processes, and the final result may approximately be described by:

$$\begin{aligned} \sigma_{xx}^2(k) &\approx (\sigma_{fn}^2 + \sigma_\nu^2)^2 (E_\varphi - \tfrac{6}{5} \\ &\times \left|\textstyle\sum_{m=-L}^{L-1} \varphi(m,0)\right|^2) \cong (\sigma_{fn}^2 + \sigma_\nu^2)^2 E_\varphi \end{aligned} \tag{6}$$

This approximation is valid for the commonly used RID distributions, which additionally satisfy the frequency marginal, since it holds: $\left|\sum_{m=-L}^{L-1} \varphi(m,0)\right|^2 \ll E_\varphi$.

3 ANALYSIS OF THE QUANTIZATION EFFECTS WITH FLOATING-POINT

In implementation based on the floating-point arithmetic the quantization only affects mantissa. Thus, in that case relative - multiplicative error appears. In order to make the appropriate analysis we will assume that, [7, 9, 12]: 1) The length of the mantissa is $(b+1)$ bits (b bits are used for the absolute value of mantissa, and one bit for sign); 2) The quantization error is a white-noise process with uniform distribution over the range -2^{-b} to 2^{-b} (mean and variance of each assumed relative error $\varsigma(n)$ are $m_\varsigma = 0$ and $\sigma_\varsigma^2 = 2^{-2b}/3 = \sigma_B^2$); 3) The error sources are uncorrelated with one another; and 4) All the errors are uncorrelated with input and consequently with all signals in the system. According to these assumptions, we will use the following model:

$$\begin{aligned} C(n,k;\varphi) = \textstyle\sum_{i=-L}^{L-1} \{ r(n,i) e^{-j4\pi ki/N} [1+\mu(n,i,k)] \\ \times \textstyle\prod_{p=1}^{L_p} [1+g(n,i,k,p)] \} \end{aligned} \tag{7}$$

$$\begin{aligned} r(n,i) = \textstyle\sum_{m=-L}^{L-1} \{ \varphi(m,i) x(n+m+i) x^*(n+m-i) \\ \times [1+e(n+m,i)][1+\rho(n+m,m,i)] \\ \times \textstyle\prod_{q=1}^{L_q} [1+d(n+m,m,i,q)] \} \end{aligned} \tag{8}$$

where $x(n) = f(n) + \epsilon(n)$. The following noise sources are introduced in the above eqs.: $\epsilon(n)$ - due to quantization of the complex input $f(n)$, $e(n+m,i)$ - due to quantization of the product $x(n+m+i)x^*(n+m-i)$, $\rho(n+m,m,i)$ - due to quantization of product of the kernel $\varphi(m,i)$ with the previous product, $\mu(n,i,k)$ - due to quantization of product of the autocorrelation function $r(n,i)$ with the basis functions $e^{-j4\pi ki/N}$. The noise sources $g(n,i,k,p)$ and $d(n+m,m,i,q)$, produced by the additions are also included. Considering the definitions and the introduced assumptions, we have:

$$2\sigma_\epsilon^2 = \sigma_e^2 = \sigma_\rho^2 = \sigma_\mu^2 = \sigma_d^2 = \sigma_g^2 = 4\sigma_B^2 = \sigma_c^2 \tag{9}$$

Suppose that the additions in our model are done by adding the adjacent elements in the first step, then the adjacent sums in the next step, and so on: $L_p = L_q = \log_2 N$. Note that the errors due to the quantization of the basic functions $e^{-j4\pi ki/N}$ and due to the kernel quantization has not been taken into analysis, because it exhibits some deterministic properties, although it can also be modeled as white noise, [9].

Since the quantization errors are small, all higher order error terms can be neglected, and the proposed model reduces to:

$$\begin{aligned} C(n,k;\varphi) = \textstyle\sum_{i=-L}^{L-1} \{ r(n,i) e^{-j4\pi ki/N} [1+\eta(n,i,k,p)] \} \\ r(n,i) = \textstyle\sum_{m=-L}^{L-1} \{ \varphi(m,i) x(n+m+i) x^*(n+m-i) \\ \times [1+\epsilon_{eq}(n+m,m,i,q)] \} \end{aligned} \tag{10}$$

where $\eta(n,i,k,p)$ and $\epsilon_{eq}(n+m,m,i,q)$ represent the equivalent noises, $\eta(n,i,k,p) = \mu(n,i,k) + \sum_{p=1}^{L_p} g(n,i,k,p)$ and $\epsilon_{eq}(n+m,m,i,q) = e(n+m,i) + \rho(n+m,m,i) + \sum_{q=1}^{L_q} d(n+m,m,i,q)$ with the corresponding variances $\sigma_\eta^2 = \sigma_\mu^2 + L_p\sigma_g^2$ and $\sigma_{eq}^2 = \sigma_e^2 + \sigma_\rho^2 + L_q\sigma_d^2$. Based on the central limit theorem, the above equivalent noises behave as Gaussian, since they represent sums of the mutually statistically independent small noises.

After some straightforward transformations we obtain the variance of the CD model in the form:

$$\begin{aligned} \sigma^2(k) = \sigma_{xx}^2(k) + \sigma_{eq}^2 \textstyle\sum_{i,m=-L}^{L-1} |\varphi(m,i)|^2 E\{|x(n+m \\ +i)|^2 |x(n+m-i)|^2\} + \sigma_\eta^2 \textstyle\sum_{i=-L}^{L} E\{|r_x(n,i)|^2\} \end{aligned} \tag{11}$$

where $\sigma_{xx}^2(k)$ is the variance of the CD's estimator when only noise $\epsilon(n)$ exist. Starting from the definitions of the noises $\eta(n,i,k,p)$ and $\epsilon_{eq}(n+m,m,i,q)$ and applying eq. (5) as well as considering the commonly used RID distributions, [1, 3, 4], satisfying the frequency marginal property, the last eq. can be simplified. Namely, knowing that, in this cases, $\varphi(m,i)$ is mainly concentrated at the origin of (m,i) plane and around i $(m=0)$ axis, [11], we have: $\sum_{m=-L}^{L-1} |\varphi(m,0)|^2 = \left|\sum_{m=-L}^{L-1} \varphi(m,0)\right|^2 = |\varphi(0,0)|^2$ for all TFDs satisfying the frequency marginal condition, where $\varphi(0,0)$ is a distribution independent constant ($\varphi(0,0)=1$). Consequently:

$$\begin{aligned}\sigma^2(k) &= (\sigma_{fn}^4+\sigma_{fn}^2\sigma_c^2)E_\varphi+\sigma_{fn}^4 \\ &\times(3+L_p+L_q)\left[E_\varphi+|\varphi(0,0)|^2\right]\sigma_c^2\end{aligned} \tag{12}$$

Note that the variance $\sigma^2(k)$ takes different values for different TFDs from the CD, depending on the factor E_φ. In [2, 6, 11] it has been shown that this factor is minimized (under the marginal conditions and time-support constraint) with the kernel of Born-Jordan TFD (B-JD) and, consequently, it can be concluded that the minimal value of the variance $\sigma^2(k)$ is obtained by B-JD.

As a criterion for qualitative comparison of the individual TFD we define the quantization noise-to-signal ratio (NSR) by:

$$NSR = (\sigma^2-\sigma^2_{|\text{without noise}})/\sigma^2_{|\text{without noise}} \tag{13}$$

where $\sigma^2_{|\text{without noise}}$ is the variance of the model assuming ideal arithmetic (the quantization errors do not exist), and $\sigma^2(k)$ is given by eq. (12). Since $|\varphi(0,0)|^2 \ll E_\varphi$, we can approximate the NSR with:

$$\begin{aligned}NSR &= \frac{\sigma_c^2}{\sigma_{fn}^2}+(3+L_p+L_q)[1+|\varphi(0,0)|^2/E_\varphi \\ &\cong \sigma_c^2/\sigma_f^2+(3+L_p+L_q)\sigma_c^2\end{aligned} \tag{14}$$

where $\sigma_f^2=\min_n\left\{\sigma_{fn}^2\right\}$ corresponds to the worst case with respect to the register length design. In this case all considered TFDs show approximately equal characteristics with respect to the NSR. The degree of the proposed approximation is different for the different TFDs and depends on the factor E_φ, [2, 6, 11]. Namely, the errors made by this approximation have been calculated in the case of most frequently used RID distributions for $N=512$ and $b=16$ (in [11], these TFDs are analyzed in details, with respect to the key factor E_φ), and it has been concluded that it ranges from the case when $|\varphi(0,0)|^2/E_\varphi = 0.0798$ for the B-JD to 0 for the Zhao-Atlas-Marks distribution (since its kernel has $\varphi(m,0)=0$, for every m, [1, 7]). Finding the $SNR[dB]$, it is shown that the maximal approximation error is 0.3916%, while the minimal approximation error (done with pseudo Wigner distribution (WD)) is 0.0263%.

Another interesting distribution which does not satisfy marginals but, in the case of multicomponent signals, may produce the sum of WDs of each component separately is the S-method, [10]. Its kernel in the time-lag domain is $\varphi(m,i)=w(m+i)w(m-i)\sin[2\pi m(2L_d+1)/N]/[(2L_d+1)K\sin(2\pi m/N)]$. For $L_d=0$ the Spectrogram follows, while for $2L_d+1=N$, we get the WD. Factor K is to keep the unbiased energy condition for any L_d. For example, for the Hanning window $w(m)$ and $L_d=4$ we get $E_\varphi=9.1104$ and $SNR[dB]=81.6509$, while for the Spectrogram ($L_d=0$) we have $SNR[dB]=81.6559$.

Substituting L_p, L_q and σ_c^2, and knowing that the duration of a TFD commonly takes an integer power of 2, $N=2^\nu$, the NSR can be represented in the form:

$$NSR \cong \frac{4}{3}(3+2\nu+1/\sigma_f^2)\cdot 2^{-2b} \tag{15}$$

Observe that the NSR consists of two parts. The first component depends only on the number of bits needed to represent mantissa, $NSR_1 \cong \frac{4}{3}(3+1/\sigma_f^2)\cdot 2^{-2b}$, and it may be easily concluded that $NSR_1[dB]$ decreases approximately $6dB$ for each bit added to the register length. The second part, $NSR_2 \cong \frac{8}{3}\nu\cdot 2^{-2b}$, is proportional to ν and, at the same time, to 2^{-2b}, i.e. quadrupling ν results in an increase in the NSR_2 which corresponds to the reduction of b by one bit.

It is interesting to present (15) as a fundamental dependence of dynamic range of the registers on the SNR:

$$b \cong 0.2075+\{10\log(3+2\nu+1/\sigma_f^2)+SNR[dB]\}/6.02 \tag{16}$$

This expression is very useful for the design of hardware for implementation of TF algorithms. Namely, it can be used to appropriately dimension registers in order to satisfy required quality, as expressed by SNR, and also to determine number of bits necessary to represent mantissa and exponent in order to find a trade-off between required accuracy and range.

4 ANALYSIS OF THE QUANTIZATION EFFECTS WITH FIXED-POINT

When the numbers are represented using fixed-point arithmetic, quantization errors occur only for multiplication. In this case, however, it is possible to cause the overflow when implementing the operation of addition. In the following analysis we will use the following model:

$$\begin{aligned}C(n,k;\varphi) &= \textstyle\sum_{i=-L}^{L-1}\{r(n,i)e^{-j4\pi ki/N}+\mu(n,i,k)\} \\ r(n,i) &= \textstyle\sum_{m=-L}^{L-1}\{\varphi(m,i)[x(n+m+i)x^*(n+m-i) \\ &\quad +e(n+m,i)]+\rho(n+m,m,i)\}\end{aligned} \tag{17}$$

Additive quantization errors stemming from this model are analogous to the ones induced by the floating-point arithmetic, [7, 5, 9, 12], with variances given by eq. (9), where $\sigma_B^2=2^{-2b}/12$.

Assume first that the analyzed signal is small enough, so that an overflow cannot occur. After several appropriate transformations (the same as ones presented in Section III) it may be shown that the model variance is:

$$\sigma^2(k)=[(\sigma_{fn}^2+\sigma_c^2/2)^2+\sigma_c^2]E_\varphi+(N^2+N)\sigma_c^2 \tag{18}$$

The above result is obtained by assuming calculations based on the conventional DFT algorithm. However, if the calculations are performed by the FFT algorithms, the results remain the same. For example, by using 'decimation-in-time' algorithm the last component from the eq. (18) is $(N-1)\sigma_\mu^2 \cong N\sigma_\mu^2$, [9].

When the analyzed signal $f(n)$ is not small enough, one should take care to prevent the overflow effects. Assuming that the samples $f(n)$ are uniformly distributed inside the interval $[0,1)$, in order to account for possible overflow, one may use one of the following methods: 1) the signal's deviding by $C = \sqrt{\sum_{i,m=-L}^{L-1} |\varphi(m.i)|}$:

$$\sigma^2(k) = \left[(\sigma_{fn}^2 + \sigma_c^2/2)^2/C^4 + \sigma_c^2\right] E_\varphi + (N^2+N)\sigma_c^2 \tag{19}$$

and 2) Using the scaling with factors 1/2 in the FFT algorithms, [9]. In that case all the signals at the input of an FFT block, generalized autocorrelation function $r(n,i)$ and the noises $e(n)$ and $\rho(n)$, get lowered by the factor of N at its output. At the same time, one should prevent an overflow in the calculation of $r(n,i)$, so that the analyzed signal is scaled by the factor $C_1 = \sqrt{\max_i \sum_{m=-L}^{L-1} |\varphi(m,i)|}$. For the commonly used RID distributions, [11], which additionally satisfy the frequency marginal, and for WD: $C_1 = \sqrt{\sum_{m=-L}^{L-1} |\varphi(m,0)|} = \sqrt{|\varphi(0,0)|} = 1$. Thus, in this case the variance (18) may be represented by:

$$\sigma^2(k) = \frac{1}{N^2}[(\sigma_{fn}^2 + \sigma_c^2/2)^2 + \sigma_c^2]E_\varphi + 5\sigma_c^2 \tag{20}$$

In these cases, according to the definition (13) and the above analysis, the NSR takes the forms:

$$NSR = \frac{\sigma_c^2}{\sigma_f^2} + \frac{\sigma_c^2}{\sigma_f^4} + \frac{(N^2+N)\sigma_c^2}{\sigma_f^4 E_\varphi} \cong \frac{N^2\sigma_c^2}{\sigma_f^4 E_\varphi} \tag{21}$$

$$NSR = \frac{\sigma_c^2}{\sigma_f^2} + \frac{C^4\sigma_c^2}{\sigma_f^4} + \frac{C^4(N^2+N)\sigma_c^2}{\sigma_f^4 E_\varphi} \cong \frac{C^4 N^2\sigma_c^2}{\sigma_f^4 E_\varphi} \tag{22}$$

$$NSR = \frac{\sigma_c^2}{\sigma_f^2} + \frac{\sigma_c^2}{\sigma_f^4} + \frac{5N^2\sigma_c^2}{\sigma_f^4 E_\varphi} \cong \frac{5N^2\sigma_c^2}{\sigma_f^4 E_\varphi} \tag{23}$$

for the conventional DFT (or FFT) and scaled FFT algorithms, respectively (with $\sigma_c^2 \ll \sigma_f^2$). For the considered TFDs, [11], the errors done by above approximations are of -0.2% and -0.002% order (for $N = 512$ and $b = 16$).

The register length, as a function of SNR and $N = 2^\nu$, can be obtained from preceding eqs. as:

$$b \cong \nu - 0.8 + \frac{SNR[dB] - 10\log(E_\varphi) - 20\log(\sigma_f^2)}{6.02} \tag{24}$$

$$b \cong \nu - 0.8 + \frac{SNR[dB] - 10\log(E_\varphi/C^4) - 20\log(\sigma_f^2)}{6.02} \tag{25}$$

$$b \cong \nu + 0.368 + \frac{SNR[dB] - 10\log(E_\varphi) - 20\log(\sigma_f^2)}{6.02} \tag{26}$$

These relations may be used for the hardware realization of TFDs, i.e. from this eqs. we can determine the necessary register's word-length for the required quality representation.

5 CONCLUSION

We have done the analysis of finite register length influence to the accuracy of results obtained by TF analysis for the cases of floating- and fixed-point arithmetic, and for the quasistationary random signals. It has been shown that commonly used TFDs from the class of the RID distributions exhibit similar performance, with respect to the SNR. The underlying theme of this paper has been the conflict between the desire to obtain fine quantization and wide dynamic range while holding the register length fixed, which may be used in the optimization of register length in hardware implementations of considered TFDs.

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