Bussgang Test: A Powerful Non–Gaussianity Test

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ABSTRACT

A process is said Bussgang if the cross–correlation function with its version passed through a zero–memory nonlinearity is proportional to the auto–correlation function of the process (invariance property). Gaussian processes are Bussgang processes too. As a consequence, Bussgangness tests may act as non–Gaussianity tests. Performance analysis shows that Bussgangness tests are more powerful than conventional Gaussian tests for correlated samples for a wide range of correlation coefficients and data lengths.

1 Introduction

Testing for deviation from Gaussianity is preliminary to other activities of signal processing, to recognize the existence of information recoverable by higher order statistics, or to detect the existence of useful signals in measurements affected by Gaussian noise.

Some classical approaches to this problem are based on frequency domain tests, which in general require large sample sets. In order to improve the detectability for relatively short signal segments, recent contributions have been focused on time domain tests. Basically, these techniques consist of measuring the distance between (theoretical) moments calculated in the Gaussian hypothesis and their estimates from sample averages after some nonlinear transformation of the series. It is known that quadratic forms built on sample deviations are asymptotically chi–square distributed under the Gaussian hypothesis. This allows to determine in principle the decision threshold for a fixed significance level (probability of Gaussian hypothesis rejection for true Gaussian series).

The most simple versions of these tests are aimed at verifying the Gaussianity of the marginal distribution of the samples. Of course, specific nonlinearities are able to detect specific deviations from Gaussianity. For instance, the third power reveals inconsistent skewness values, whereas the fourth power reveals anomalies of the kurtosis. Non-Gaussian distributions having skewness and kurtosis close to the values pertaining to the Gaussian case are undetectable by these tests.

Likewise, the (complex) exponential nonlinearities employed in the characteristic function oriented tests [1, 2] may be unable to detect some other specific non–Gaussian behaviors.

In order to deal with as many cases as possible in practical situations where the nature of the measured samples is totally unknown, composite tests based on sets of nonlinearities have been proposed and characterized in the recent literature. The higher–order moment approach is based on the joint use of multiple moments [3], whereas the empirical characteristic function approach employs different values of the parameter in the exponential. In some cases, such as linear filtered signals, the non–Gaussian nature is difficult to detect from the analysis of the marginal distribution [8]. For this reason, multivariate detectors have been proposed.

A unified theory of time domain Gaussianity tests based on finite memory nonlinearities has been very recently exposed in [4]. It is based on Price theorem, which relates the moments of Gaussian n–variates to the moments of given functions of these variates. This approach enlightens how tests might be designed in general. Specifically, higher order moment based tests and empirical characteristic function based tests are derived for the n–variate case.

In this contribution, we propose an alternative approach based on the Bussgang property. Even though Bussgang theorem is implied by Price theorem, it presents peculiar aspects which makes it particularly attractive for various applications of signal processing. For instance, it constitutes the principle underlying some well known techniques of blind deconvolution employed in data communication systems and in geophysics. Some preliminary results were presented by the authors in [7]. Here, we develop a procedure for deciding whether a finite segment of a signal can be considered as a realization of a Bussgang process or not. If the Bussgang hypothesis is rejected, the process must be (by definition) non–Gaussian too. As a consequence, the Bussgangness test acts as a non–Gaussianity test.

2 The Bussgang test

As well known [7], Bussgang theorem states that the cross-correlation function of a Gaussian stationary process $x[n]$ and of its version passed through a zero-memory nonlinearity $g(\cdot)$ is proportional to the auto-correlation function of the process, namely

$$R_{xg}[k] = k_g \cdot R_{xx}[k]$$

having defined the correlation functions

$$R_{xg}[k] = \mathrm{E}\{x[n+k] \cdot g(x[n])\}$$
$$R_{xx}[k] = \mathrm{E}\{x[n+k] \cdot x[n]\}$$

The proportionality factor k_g depends on the nonlinearity $g(\cdot)$ and it can be expressed as [5]

$$k_g = \mathrm{E}\{\dot{g}(x[n])\}$$

This property has been extended to complex process in [6, 9] and generalized to the multivariate case in [10].

Conventional Gaussianity tests based on Price's theorem consist of measuring the deviation from proportionality of the sample auto-correlation to the sample cross-correlation for a given nonlinearity, with the theoretical proportionality factor. A set of non-Gaussianity errors $e_G(i)$ can be accordingly defined as:

$$e_G[i] = R_{xg}[i] - k_g \cdot R_{xx}[i]$$

The most common conventional test uses to the correlations estimated for $i=0$, *i.e.*:

$$e_G[0] = R_{xg}[0] - k_g \cdot R_{xx}[0] \quad (1)$$

The estimation of k_g (which is *independent* of i) from a small amount of data is a critical point of this procedure, since it is a known function (depending on the employed nonlinearity) of the estimated power $R_{xx}[0]$. As a consequence, the error in the estimate of $R_{xx}[0]$ affects twice (1).

An alternative procedure consists of defining a set of errors $e_B[i]$ depending on the ratio of two equations derived from the Bussgang theorem, *i.e.*:

$$e_B[i] = \frac{R_{xg}[i]}{R_{xg}[0]} - \frac{R_{xx}[i]}{R_{xx}[0]} \quad (2)$$

In such a case, this version of the Bussgang test does not depend on the (estimated) proportionality factor k_g.

The general multivariate test can be accomplished as follows [3]. Collect a number M of testing variables (errors)

$$\mathbf{e} = \Big[e_B[0] \ \cdots \ e_B[M-1]\Big]^{\mathrm{T}}$$

and consider their covariance matrix

$$\Sigma_{\mathrm{e}} = \mathrm{E}\left\{\Big[\mathbf{e} - \mathrm{E}\{\mathbf{e}\}\Big] \cdot \Big[\mathbf{e} - \mathrm{E}\{\mathbf{e}\}\Big]^{\mathrm{T}}\right\}$$

Since the errors are asymptotically normal distributed, the Bussgangness test is formulated as the following binary hypothesis test

$$H_0: \quad \mathbf{e} \sim \mathcal{N}\left(\mathbf{0}, N^{-1}\Sigma_{\mathrm{e}}\right) \qquad \text{(Bussgang)}$$

$$H_1: \quad \mathbf{e} \sim \mathcal{N}\left(\mathrm{E}\{\mathbf{e}\}, N^{-1}\Sigma_{\mathrm{e}}\right) \qquad \text{(non}-\text{Bussgang)}$$

Under the hypothesis H_0 the sum of the squares of testing variables $\mathbf{e}$ is asympototically chi-square distributed [3], *i.e.* it results for $N \to \infty$

$$d \stackrel{\text{def}}{=} N \cdot \mathbf{e}^{\mathrm{T}}\Sigma_{\mathrm{e}}^{-1}\mathbf{e} \sim \chi^2_M$$

Therefore, for a fixed α-level of significance, the test reduces to a chi-square test:

$$d \underset{H_0}{\overset{H_1}{\gtrless}} t_G = \chi^2_M(\alpha)$$

The thresold t_G is found using standard χ^2 table [11] after fixing the probability of false alarm

$$P_{\mathrm{F}} \stackrel{\text{def}}{=} \alpha \leq P\left(d \geq \chi^2_M | H_0\right)$$

Let us note that the test is intrinsically multivariate; its dimensionality depends on how many correlation lags are included. Moreover, it is easily verified that this approach generates well defined higher order moment-based tests. However, care must be taken in selecting the correlation lags to be included in the test since the variance of these latter can significantly vary from lag to lag and this can significantly affect the overall performance of the multivariate test which may result less powerful than a more simple scalar (single lag) test.

To assess the applicability of the Bussgang approach in comparison to conventional Gaussianity test, in the following we will consider only scalar tests.

3 Simulation results

Let us focus our attention on the behavior of the conventional Gaussian and of the Bussgang tests, by comparing their performance in detecting the presence of a correlated non-Gaussian process embedded in additive Gaussian noise. The analysis is accomplished for several nonlinearities, derived from the existing tests, according to the following table

Test	$g(x)$
characteristic function [1, 2]	$\frac{1-\cos\omega x}{x}$
fourth-order moment [3]	x^3
signum [7]	$\operatorname{sign} x$

Table 1: *Nonlinearities employed in the Bussgang test.*

In particular, we have evaluated the significance level (*i.e.* the probability of rejecting the Gaussian hypothesis when the signal is actually Gaussian) and the power of the various tests (*i.e.* the probability of rejecting the Gaussian hypothesis when the signal is not actually Gaussian).

A crucial point is the analysis of the effect of the correlation coefficient of the process under examination since the Bussgang only applies to colored data by its nature. In fact, it tests the estimated correlation coefficients of auto- and cross-correlation functions. Therefore, we have tested colored signals obtained from zero-mean, white binary signals, filtered by a single pole IIR filter and corrupted by Gaussian white noise ($\mathrm{SNR} = 20\mathrm{dB}$).

The performance of the conventional and the Bussgang tests has been evaluated considering several values of the correlation coefficient in the range $(0.5 < \rho < 0.9)$. The percentage of the rejection of the Gaussian hypothesis has been estimated through 1000 Monte Carlo trials, after fixing all the employed thresholds in order to obtain the same significance level 5% in all cases. The value $\omega = 1$ has been adopted for the characteristic function based test.

The estimated power of the conventional Gaussian test (1) and the Bussgang test (2) are shown in the figs.1-3 for the three considered nonlinearities for $(0.5 < \rho < 0.8)$ and an observation window $N = 128$ samples. The same results for $(0.6 < \rho < 0.9)$ and an observation window of $N = 256$ samples are reported in the figs.4-6.

4 Conclusion

We have developed a procedure for deciding whether a finite segment of a signal can be considered as a realization of a Bussgang process or not. The approach is based on Bussgang invariance property of stationary random process which relates the auto-correlation to the cross-correlation involving a nonlinear version of the process itself.

A binary hypotheses test is described to detect the possible Bussgang's invariance property of the cross-correlation or, equivalently, of the underlying auto-regressive model coefficients.

The usefulness of the Bussgang test is found when the sample is drawn form from non-white stationary processes.

The results show that the Bussgang test performs (to detect non-Gaussian signals) even better than a conventional Gaussian test itself for a wide range of correlation coefficients ρ and data lengths. This difference appears particularly significant for the all the herein analyzed nonlinearities.

In particular, the performance of the cubic-based test (figs.1 and 4) is strongly enhanced with respect to the kurtosis-based scalar test [3].

In summary, Bussgang test is not only an operative procedure to assess the possible Bussgang's invariance property of the cross-correlation and the AR model coefficients, but it also acts as a powerful non-Gaussianity test in the presence of correlated input samples.

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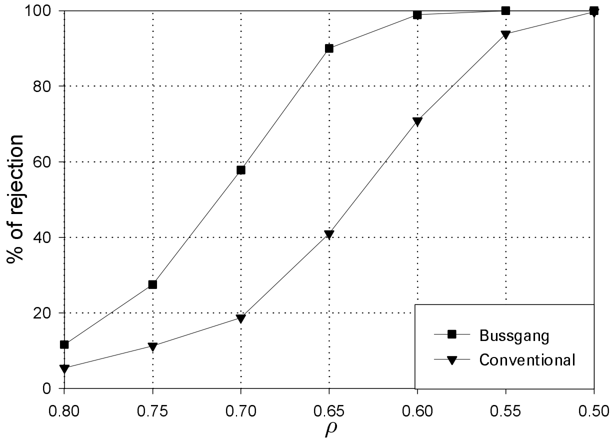

Figure 1: *Percentage of rejection of the Gaussian hypothesis vs the correlation coefficient ρ using a cubic nonlinearity and $N = 128$ samples.*

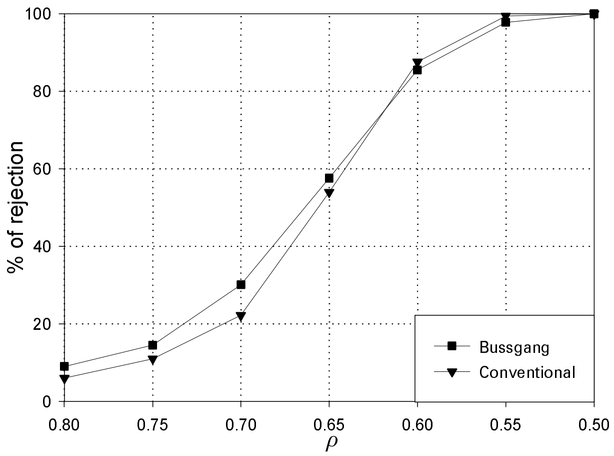

Figure 2: *Percentage of rejection of the Gaussian hypothesis vs the correlation coefficient ρ using a cosine nonlinearity and $N = 128$ samples.*

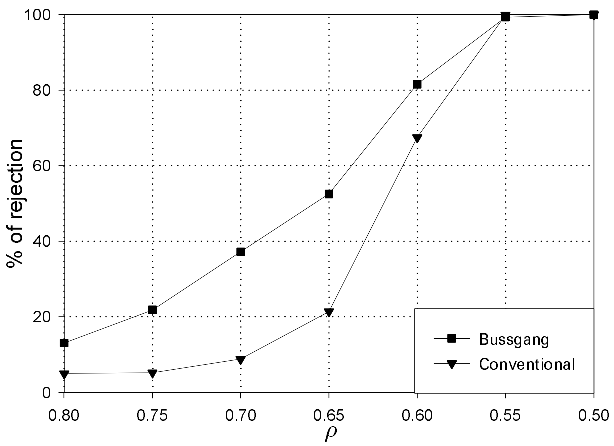

Figure 3: *Percentage of rejection of the Gaussian hypothesis vs the correlation coefficient ρ using a signum nonlinearity and $N = 128$ samples.*

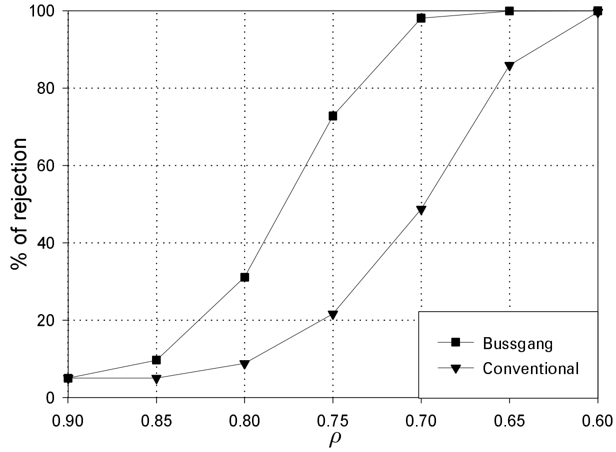

Figure 4: *Percentage of rejection of the Gaussian hypothesis vs the correlation coefficient ρ using a cubic nonlinearity and $N = 256$ samples.*

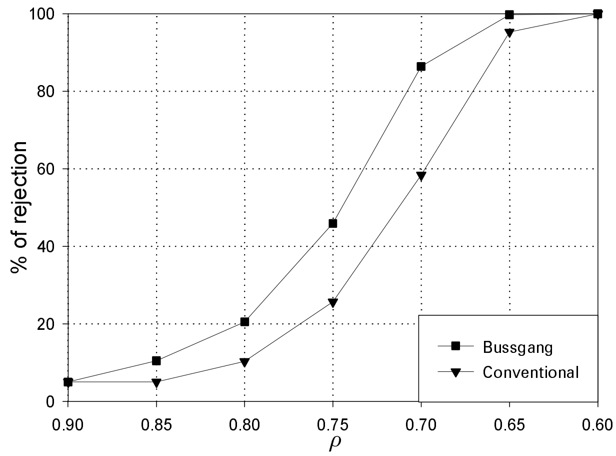

Figure 5: *Percentage of rejection of the Gaussian hypothesis vs the correlation coefficient ρ using a cosine nonlinearity and $N = 256$ samples.*

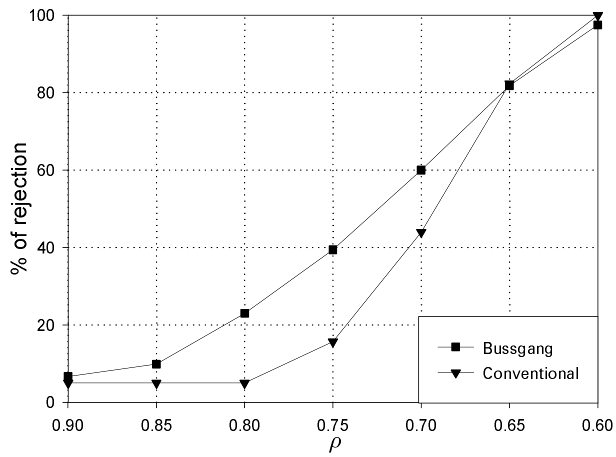

Figure 6: *Percentage of rejection of the Gaussian hypothesis vs the correlation coefficient ρ using a signum nonlinearity and $N = 256$ samples.*