# A CONSTANT MODULUS ARRAY FOR REAL SIGNALS

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**ABSTRACT**

A modified Constant Modulus Algorithm (CMA) is proposed for real signals impinging upon an array. The algorithm solves the *mix-up* problem of CMA which occurs when real signals propagate through complex channels. Moreover, it decreases computational complexity and extends the maximum number of real sources which can be resolved by a given array. Simulations are presented to support the analysis.

## 1 INTRODUCTION

The Constant Modulus (CM) algorithm [1, 2] was first applied to array processing [3] to reconstruct transmitted signals from their received mixtures. The sequential and blind nature of the CM array has many advantages over block-based and parametric-array processing. For example, it does not need precise array calibration as generally required by parametric-array processing algorithms. Moreover, in an urban or hilly-rural terrain, where it is normal to find that two or more signals are highly correlated, or coherent, conventional direction finding algorithms based upon subspace methods such as MUSIC [4] and ESPRIT [5] fail; whereas the CM array remains applicable because it does not depend upon the actual Directions Of Arrival (DOA) of the signals. The CM array removes coherent signals as the linear-combination of their DOA, i.e., their effective DOA [6]. Although, an algorithm has been proposed in [7], which reduces array calibration and coherent effects, it is a block-based algorithm which implies that its ability to track moving sources is restricted.

However, because of some undesirable characteristics of CMA, the CM array still has problems which many researchers have been addressing. Signal cancellers were introduced in [8] in order to prevent two or more CM arrays from capturing the same signal, and have been modified in [9] to remove both a captured signal and its delayed-versions. Local minima and convergence rate problems have been addressed by many workers. A variable stepsize in [10] allows CMA to track the same signals in a fast-fading environment. To increase selectivity of the equaliser, a spatial-temporal model has been studied [11, 12]. Generally these modifications make the CM array smarter, whilst requiring more computational complexity.

In [13], another problem of CMA has been pointed out. The problem will occur when real modulated signals, such as Amplitude-Shifted-Keying (ASK) signals, propagate though complex channels. The real and the imaginary parts of the equaliser output will usually *mix-up* two different delayed-versions of the transmitted signal, for which the Signal to Interference and Noise Ratio (SINR) will be low in the single-user case. The problem is obviously more severe in the multi-user case, but has not been explained in [13].

In this paper, we will propose a modified CMA algorithm to solve the *mix-up* problem in section 2. We also show that the modification requires less computational complexity than the conventional CMA. Moreover, the modification will be proven to handle a greater number of signals than the number of sensors in section 3. Then we will further demonstrate the mix-up effect in the multi-user case by simulations in section 4. These results evidence the advantage of the proposed modification. Finally, discussions and conclusions will be presented in section 5.

## 2 REAL CMA

Godard's cost function [1] for the CM algorithm can be written as

$$J_{Godard} \triangleq E\{(|\underline{\mathbf{w}}^H \underline{\mathbf{x}}|^p - R_p)^2\} \tag{1}$$

where $\underline{\mathbf{w}}$ is the complex adaptive weight vector, $\underline{\mathbf{x}}$ is the complex input mixture, $R_p = E\{|s^{2p}|\}/E\{|s^p|\}$ is called the dispersion constant, $s$ is the transmitted source, $p$ is a positive-integer constant, and $E\{\cdot\}$ and $(\cdot)^H$ denote respectively statistical expectation and Hermitian transpose operations.

With the assumption that all the signals, impinging upon the array, are real, we propose to modify the cost function to

$$J' \triangleq E\{(|Re(\underline{\mathbf{w}}^H \underline{\mathbf{x}})|^p - R_p)^2\} \tag{2}$$

where $Re(\cdot)$ denotes the real part of $(\cdot)$.

In order to find the optimal solution for $\underline{\mathbf{w}}$, we minimise the cost function with respect to $\underline{\mathbf{w}}$.

$$\begin{aligned}\nabla_{\underline{\mathbf{w}}} J' &= 4pE\{(|Re(\underline{\mathbf{w}}^H\underline{\mathbf{x}})|^p - R_p)\cdot \\ &\quad (|Re(\underline{\mathbf{w}}^H\underline{\mathbf{x}})|^{p-1})\cdot \frac{\partial|Re(\underline{\mathbf{w}}^H\underline{\mathbf{x}})|}{\partial\underline{\mathbf{w}}^*}\} \qquad (3)\end{aligned}$$

By the chain rule and $\frac{\partial Re(\underline{\mathbf{w}}^H\underline{\mathbf{x}})}{\partial\underline{\mathbf{w}}^*} = \frac{\underline{\mathbf{x}}}{2}$ [14],

$$\begin{aligned}\frac{\partial|Re(\underline{\mathbf{w}}^H\underline{\mathbf{x}})|}{\partial\underline{\mathbf{w}}^*} &= \frac{\partial|Re(\underline{\mathbf{w}}^H\underline{\mathbf{x}})|}{\partial Re(\underline{\mathbf{w}}^H\underline{\mathbf{x}})}\frac{\partial Re(\underline{\mathbf{w}}^H\underline{\mathbf{x}})}{\partial\underline{\mathbf{w}}^*} \qquad (4)\\ &= \frac{Re(\underline{\mathbf{w}}^H\underline{\mathbf{x}})}{|Re(\underline{\mathbf{w}}^H\underline{\mathbf{x}})|}\frac{\underline{\mathbf{x}}}{2}, \quad Re(\underline{\mathbf{w}}^H\underline{\mathbf{x}}) \neq 0 (5)\end{aligned}$$

Then substitute (5) in (3), to obtain

$$\begin{aligned}\nabla_{\underline{\mathbf{w}}} J' &= 2pE\{(|Re(\underline{\mathbf{w}}^H\underline{\mathbf{x}})|^p - R_p)\cdot \\ &\quad |Re(\underline{\mathbf{w}}^H\underline{\mathbf{x}})|^{p-2}\cdot Re(\underline{\mathbf{w}}^H\underline{\mathbf{x}})\cdot\underline{\mathbf{x}}\} \qquad (6)\end{aligned}$$

Adaptation of the weight vector in a form of a steepest-descent search can be written as

$$\underline{\mathbf{w}}_{n+1} = \underline{\mathbf{w}}_n - \mu\nabla_{\underline{\mathbf{w}}} J'|_{\underline{\mathbf{w}}=\underline{\mathbf{w}}_n} \qquad (7)$$

where $\mu$ is the stepsize of the adaptation.
Finally, a stochastic gradient algorithm can be obtained by dropping the expectation operator and replacing variables by their instantaneous values, that is

$$\begin{aligned}y_n &= Re(\underline{\mathbf{w}}_n^H\underline{\mathbf{x}}_n) \qquad (8)\\ e_n &= p(|y_n|^p) - R_p)|y_n|^{p-2}y_n \qquad (9)\\ \underline{\mathbf{w}}_{n+1} &= \underline{\mathbf{w}}_n - 2\mu e_n\underline{\mathbf{x}}_n \qquad (10)\end{aligned}$$

where $y_n$ is the output of the CM array.

Equations (8), (9) and (10) are similar to Godard's equations except that we incorporate explicitly real knowledge in the modified CM algorithm. Notice that we do not have to find the imaginary result of the complex multiplication in (8), moreover (9) is an equation containing only real values. If $p = 2$, this modified CM algorithm requires $3M+3$ real multiplications and $2M$ real additions per iteration, whereas the conventional algorithm requires $8M+10$ real multiplications and $8M+3$ real additions per iteration, where $M$ is the number of sensors in the beamformer or length of the equaliser. Hence, computational complexity of the modified CM algorithm is less than half of that of the conventional one.

## 3 APERTURE EXTENSION

If there are $N$ uncorrelated sources impinging upon an array of $M$ sensors, a well-known limitation of the conventional CM array is that $N$ must not be greater than $M$, in order to reconstruct all sources. However, if all the sources are known to be real, we can incorporate this information by applying the modified CMA to beamformers. We must establish how many sources the beamformers can process.

Given that there are $2M$ real sources, $\underline{\mathbf{s}}$, impinging upon $M$ sensors, the measurement signals, $\underline{\mathbf{x}}$, can be written as

$$\underline{\mathbf{x}} = \mathbf{A}\underline{\mathbf{s}} + \underline{\mathbf{n}} \qquad (11)$$

where $\mathbf{A}$ is the complex $M\times 2M$ steering matrix and $\underline{\mathbf{n}}$ is the complex measurement noise vector, in which each element is assumed to be zero-mean, white and uncorrelated with the received signals and all other noise elements.

Without loss of generality, we assume that the modified CM arrays try to find a $2M\times M$ weight matrix, $\mathbf{W} = [\underline{\mathbf{w}}_1^H;\ldots;\underline{\mathbf{w}}_{2M}^H]$, where each weight vector, $\underline{\mathbf{w}}_i$, $i = 1,\ldots,2M$ captures the $i$th source. By (8) and (11)

$$\begin{aligned}\underline{\mathbf{y}} &= Re(\mathbf{W}\underline{\mathbf{x}}) \qquad (12)\\ &= Re(\mathbf{W}(\mathbf{A}\underline{\mathbf{s}}+\underline{\mathbf{n}})) \qquad (13)\\ &= Re(\mathbf{W}\mathbf{A})\underline{\mathbf{s}} + Re(\mathbf{W}\underline{\mathbf{n}}) \qquad (14)\end{aligned}$$

Clearly, the output $\underline{\mathbf{y}} \cong \underline{\mathbf{s}}$ when

$$Re(\mathbf{W}\mathbf{A}) = \mathbf{I}_{2M} \qquad (15)$$

and $Re(\mathbf{W}\mathbf{A})\underline{\mathbf{s}} \gg Re(\mathbf{W}\underline{\mathbf{n}})$, where $\mathbf{I}_{2M}$ is a $2M\times 2M$ identity matrix.

Similarly, it is evident that the conventional CM array cannot find the optimum solution, when the noise is zero, which is

$$\mathbf{W}\mathbf{A} = \mathbf{I}_{2M} \qquad (16)$$

because this set of $2M\times 2M$ effective scalar equations has only $2M\times M$ unknowns ($w_{ji}^*$, $i = 1,\ldots,M$, $j = 1,\ldots,2M$ and $(\cdot)^*$ denotes complex conjugate.), i.e., an over-determined set of equations.

Equation (15) neglects the imaginary path of the output signal, and can be written as

$$\mathbf{W}_r\mathbf{A}_r - \mathbf{W}_i\mathbf{A}_i = \mathbf{I}_{2M} \qquad (17)$$

where $(\cdot)_r$ and $(\cdot)_i$ denote respectively the real and the imaginary parts of $(\cdot)$. This equation can be written in the matrix form as

$$(\mathbf{W}_r\mathbf{W}_i)\cdot\begin{pmatrix}\mathbf{A}_r\\ -\mathbf{A}_i\end{pmatrix} = \mathbf{I}_{2M} \qquad (18)$$

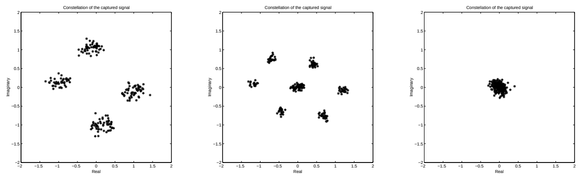
Because both $[\mathbf{W}_r\mathbf{W}_i]$ and $[\mathbf{A}_r - \mathbf{A}_i]$ are $2M\times 2M$ matrices, so (14) is an exactly-determined equation, hence the modified CMA can converge to this exact solution.

In the under-determined case, i.e., $N < 2M$, the modified CMA is also able to find an optimum solution. Intuitively, if $N \leq 2M$, the minimum norm solution which the modified CMA can find is

$$(\mathbf{W}_r\mathbf{W}_i) = \begin{pmatrix}\mathbf{A}_r\\ -\mathbf{A}_i\end{pmatrix}^\sharp \qquad (19)$$

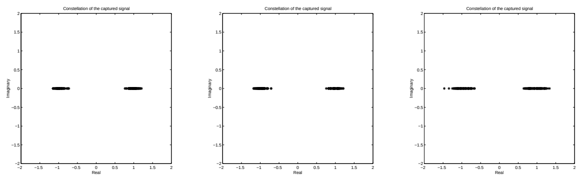
where $(\cdot)^\sharp$ denotes pseudoinverse operation.

The above analysis approves that the number of sources can be extended to twice the number of sensors, when all sources are real and the modified CMA is applied to the beamformers.



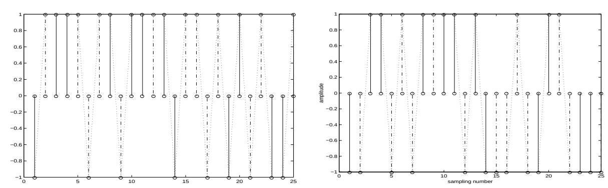
(a) CM array ♯1 (b) CM array ♯2 (c) CM array ♯3

Figure 1: Constellation of the captured signal by the conventional CM array



(a) CM array ♯1 (b) CM array ♯2 (c) CM array ♯3

Figure 2: Constellation of the captured signal by the modified CM array



(a) Real + Imaginary (b) Real - Imaginary

Figure 3: Reconstruction of two signals (dotted line) from the real (solid line) and imaginary (dash-dotted line) parts of the output of the first CM array

## 4 COMPUTER SIMULATIONS

### 4.1 Mix-up Effect

This experiment is set up to show the mix-up effect in the multi-user case (without InterSymbol Interference (ISI)) of the conventional CM array and to confirm that the modified CMA can solve the problem.

Suppose a linear array has three omnidirectional sensors at position 0, 1 and 2 ($\lambda/2$ units are employed). Three uncorrelated BPSK sources ($\pm 1$) illuminate the array from DOAs of $-10°, 0°$ and $15°$ to the broadside of the array. Additive white noise is at 20 dB level. The conventional and modified CMA will be applied to a beamformer as sequential schemes [6] in order to reconstruct these sources. All initial values of the weight vector of CMA 2-2 [2] are $[1\ 0\ 0]^T$.

Figure 1(a) shows that the first stage of the CM array using conventional CMA does not provide a real output but mixes up 2 input signals and yields a signal which takes the form of a QPSK signal. The information of both input signals is mixed in the real and the imaginary parts of the output. One signal can be reconstructed from the real part *plus* the imaginary part of the output as shown in Figure 3(a). The other can be obtained from the real part *minus* the imaginary part of the output as shown in Figure 3(b). Certainly, the first signal canceller cannot know that its input is the mixture of two signals. It does cancel its reference input, which is a QPSK signal, from the input mixture formed from three BPSK signals. Hence its output mixture is of the form of one QPSK signal and one BPSK signal. Then, the second CM array will again mix-up the remaining two signals. The constellation of the mix-up of BPSK and QPSK signals has the form of the hexagon in Figure 1(b). The third signal is certainly not computable from this mix-up. The hexagonal signal carries the same information as the input mixture does, therefore the output mixture of the second signal canceller contains no information. This is confirmed by the output of the third CM array in Figure 1(c).

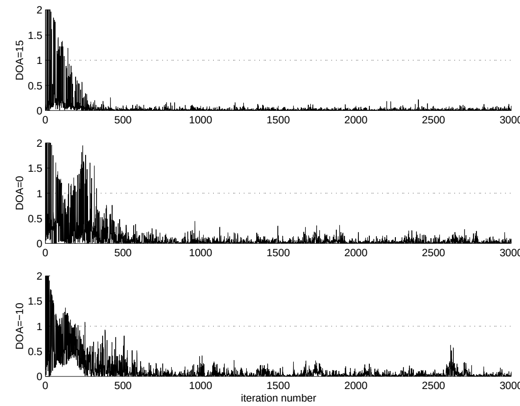



Figure 4: Squared error of output from the real CM arrays and their corresponding transmitted sources.

On the other hand, the output of the three modified CM arrays are all open-eyed BPSK signals as shown in Figure 2. They are tested to have the same information as their corresponding transmitted signal as shown in Figure 4. Hence the results support the objective of the proposed modification.

Note that the outputs of the conventional arrays depend upon the number of signals, the number and DOAs of the transmitted signals, and may have different constellations from the example in the experiment.

### 4.2 Aperture of the Modified CM Array

To confirm that the modified CMA can handle the number of signals up to twice the number of sensors, the following experiment is set up. The array, type of the noise and transmitted signals are the same as those in the previous experiment. But there are six transmitted signals in this experiment. The sources are from direction of $-50°$, $-30°$, $-10°$, $20°$ $40°$ and $60°$ angles to the broadside of the array. All weight vectors of CMA 2-2 are initialised with $[1, 0, \ldots, 0]^T$.

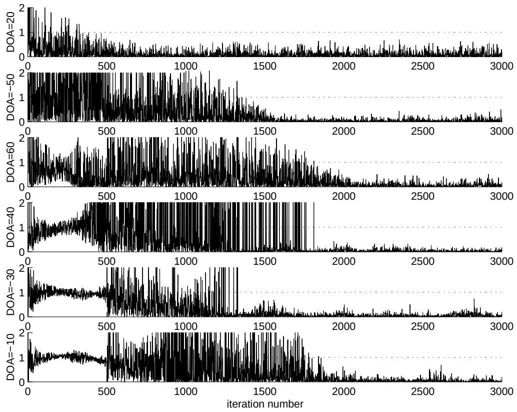



Figure 5: Squared error of output from the modified CM arrays and their corresponding transmitted sources.

The results in Figure 5 show that all squared differences between the output of the modified CM array and the transmitted signals converge below the threshold (dotted line), which confirm the analysis which proved that the modified CM array can operate with the number of signals equal to twice the number of sensors.

## 5 DISCUSSIONS & CONCLUSIONS

Complex-modulated signals are more efficient than real-modulated signals in term of channel capacity, and are normally selected to be used for transmission. However, this work has followed [13] who studied the special case when real signals travel though complex channels.

Adaptation of the weight vectors in a complex form with the conventional CM algorithm will usually cause the mix-up effect. We have proposed a modified CM algorithm to solve this problem. Incorporating real knowledge of the sources, implies that $y_n$ and the effective error within the adaptation are real, and prevents the algorithm from mixing up on two sources.

The number of operations required in the modified algorithms is one half of the conventional CM algorithm. Furthermore, real adaptation has twice the number of degrees of freedom. Therefore the maximum number of resolvable sources, which can impinge upon an array of $M$ sensors, is extended to $2M$. Simulations have been presented to support the assumption of the proposed modification and confirm the analysis work.

The error performance surface of this algorithm has to be investigated in future work. Modification of the cost function to make the algorithm smarter in time-varying channels is currently being undertaken.

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