INTER-RELATIONSHIPS BETWEEN DIFFERENT STRUCTURES FOR PERIODIC SYSTEMS

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ABSTRACT

A self-contained, clear and consistent notation is developed for five different mathematical models of linear periodically time-varying (LPTV) filters. Then the various inter-relationships between the different structures are derived.

1 INTRODUCTION

Some recent applications of LPTV filters and systems include bandwidth compression [1], transmultiplexing [2], speech [3] and image [4] scrambling, communications [5], and limit-cycle removal in digital filters [6]. More recently, the theory and application of two-dimensional (2-D) LPTV systems have also received some attention [4, 6-10].

Five popular representations of LPTV systems (see Figs 2-5) have emerged [2,3,7,8,11-14]: *the LPTV difference equation, multiple-input/multiple-output, multiple-input/single-output, single-input/multiple-output, and the modulator filter*.

One of the problems of dealing with any of the above is the wide range of nomenclatures and descriptions. Generally all five realisations are not considered together. In addition, the models sometimes only refer to FIR structures, or are very general schematics not showing the derivations or the various inter-relationships.

2 THE VARIOUS EQUIVALENT STRUCTURES

Linear Difference Equation

Consider the following general LPTV filter:

$$y(n)=\sum_{i=0}^{M_1} a_i(n)x(n-i)+\sum_{j=1}^{M_2} b_j(n)y(n-j) \qquad (1)$$

where $\quad a_i(n+N_i)=a_i(n), b_j(n+N_j)=b_j(n)$. (2)

The fundamental period is N, where N is the lowest common multiple of $\{\{N_i\}_{i=0}^{M_1},\{N_j\}_{j=1}^{M_2}\}$. We may now re-write (1) using a set of <u>time-invariant</u> coefficients to give:

$$\begin{aligned} y_k(n) &= \sum_{i=0}^{M_1} a_{ik}x_{<k-i>}(n+[k-i]) \\ &\quad +\sum_{j=1}^{M_2} b_{jk}y_{<k-j>}(n+[k-j]) \end{aligned} \qquad (3)$$

$$\left.\begin{aligned} x_k(n) &= x(nN+k) \\ y_k(n) &= y(nN+k) \\ a_{ik} &= a_i(k)=a_i(nN+k) \\ b_{jk} &= b_j(k)=b_j(nN+k) \end{aligned}\right\} \begin{aligned} &-\infty\le n\le\infty \\ &0\le k\le N-1 \\ &0\le i\le M_1 \\ &1\le j\le M_2 \end{aligned} \qquad (4)$$

with $<\bullet>$ and $[\bullet]$ relating to arithmetic modulo-N[1].

Green's Function

Another description is based upon the following general expression for a linear time-varying system:

$$y(n)=\sum_{\ell=-\infty}^{\infty} h(n,\ell)x(\ell)=\sum_{\ell=-\infty}^{\infty} x(\ell)c_n(n-\ell)=\sum_{\ell=-\infty}^{\infty} x(\ell)r_\ell(n-\ell) \quad (5)$$

where $h(n,\ell)$ is the Green's function of the system, and is the response of the filter at time 'n' due to a unit impulse applied at time 'ℓ'. We can also define:

$$\begin{aligned} &c_k(\ell)=r_{k-\ell}(\ell)=h(k,k-\ell), r_k(n)=c_{k+n}(n)=h(k+n,k), \\ &c_n(n-\ell)=r_\ell(n-\ell)=h(n,\ell). \end{aligned} \qquad (6)$$

where $c_k(\ell)$ and $r_k(n)$ represent, respectively, the system response at time 'k' due to a unit impulse applied 'ℓ' samples earlier, and the response at time '$k+n$' due to a unit impulse applied 'n' samples earlier. Importantly, different nomenclature is now used from previous publications [2,3,11-14] to provide intuitive understanding. Here $c_k(\ell)$ is the ℓ-th element of the k-th **<u>column</u>** of $h(n',\ell')$, and $r_k(n)$ is the n-th element of the k-th **<u>row</u>** of $h(n',\ell')$, with columns and rows zero-referenced from the main diagonal of $h(n',\ell')$, as shown in Fig.1. Using $c_k(\ell)$ and $r_k(n)$ avoids the nomenclature problems of taking the z-transform of $h(k,k-\ell)$ and $h(k+n,k)$.

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[1] Let $X=mN+n, 0\le n\le N-1$, where X, m, n and N are all integers, with N and n non-negative. Then we can define: $<X>=n$, and $[X]=m$ for modulo-N arithmetic.

Now for a LPTV system with period N, it is not difficult to show that the following relationships also hold:

$$h(n,\ell)=h(n+N,\ell+N),\ c_k(\ell)=c_{k+N}(\ell),\ r_k(n)=r_{k+N}(n) \quad (7)$$

This result will be used in subsequent analysis. Clearly, there are now only N unique rows and N unique columns in $h(n,\ell)$, and arithmetic mod-N[1] should be appropriately used in all the previous expressions giving:

$$y(n)=\sum_{\ell=-\infty}^{\infty} x(\ell)c_{<n>}(n-\ell)=\sum_{\ell=-\infty}^{\infty} x(\ell)r_{<\ell>}(n-\ell) \quad (8)$$

$$\begin{aligned} c_k(\ell)&=r_{<k-\ell>}(\ell)=h(k,k-\ell)\\ r_k(n)&=c_{<k+n>}(n)=h(k+n,k)\\ 0&\le k\le N-1\end{aligned} \quad (9)$$

Multiple-Input/Multiple-Output Model

This follows a similar approach as previously used for 2-D [8]. The objective here is to construct an equivalent linear time-invariant multiple-input/multiple-output (MIMO) model for (1). Now (3), which is an equivalent representation of (1), defines N LTI linear difference eqns (one for each of the N different values of k). Taking the z-transform of each one of these linear difference eqns results in N linear simultaneous eqns in the N z-transforms of the sub-sampled outputs of $y(n)$, i.e. $\{Y_k(z)\}_{k=0}^{N-1}$. These simultaneous eqns can now be solved to give the following MIMO model in Fig.2:

$$\boldsymbol{Y}(z)=\boldsymbol{H}(z)\boldsymbol{X}(z) \quad (10)$$

where

$$\boldsymbol{Y}(z)=[Y_0(z)Y_1(z)\cdots Y_{N-1}(z)]^t$$

$$\boldsymbol{X}(z)=[X_0(z)X_1(z)\cdots X_{N-1}(z)]^t$$

$$\boldsymbol{H}(z)=\begin{bmatrix} H_{00}(z) & H_{01}(z) & \cdots & H_{0,N-1}(z)\\ H_{10}(z) & H_{11}(z) & \cdots & H_{1,N-1}(z)\\ \vdots & \vdots & \ddots & \vdots\\ H_{N-1,0}(z) & H_{N-1,1}(z) & \cdots & H_{N-1,N-1}(z)\end{bmatrix} \quad (11)$$

$y_k(n)\leftrightarrow Y_k(z),\ 0\le k\le N-1$, etc.

$$\boxed{H_{k\ell}(z)=f_{k\ell}\left(\{a_{ij}\}_{i,j=0,0}^{M_1,N-1},\{b_{ij}\}_{i,j=0,0}^{M_2,N-1}\right)} \quad (11a)$$

Example

Consider (1), where $M_1=M_2=N=2$. Taking the z-transform of (3) for k=0,1 gives

$$\begin{aligned}&\begin{bmatrix}1-b_{20}z^{-1} & -b_{10}z^{-1}\\ -b_{11} & 1-b_{21}z^{-1}\end{bmatrix}\begin{bmatrix}Y_0(z)\\ Y_1(z)\end{bmatrix}\\ &=\begin{bmatrix}a_{00}+a_{20}z^{-1} & a_{10}z^{-1}\\ a_{11} & a_{01}+a_{21}z^{-1}\end{bmatrix}\begin{bmatrix}X_0(z)\\ X_1(z)\end{bmatrix}\end{aligned} \quad (12)$$

and solving (12) we get (10), where for $\boldsymbol{H}(z)$:

$$\begin{aligned} H_{00}(z)&=[a_{00}+(a_{20}-a_{00}b_{21}+a_{11}b_{10})z^{-1}-a_{20}b_{21}z^{-2}]/D(z)\\ H_{01}(z)&=[z^{-1}(a_{10}+a_{01}b_{10})+(a_{21}b_{10}-a_{10}b_{21})z^{-2}]/D(z)\\ H_{10}(z)&=[(a_{11}+a_{00}b_{11})+(a_{20}b_{11}-a_{11}b_{20})z^{-1}]/D(z)\\ H_{11}(z)&=[a_{01}+(a_{21}-a_{01}b_{20}+a_{10}b_{11})z^{-1}-a_{21}b_{20}z^{-2}]/D(z)\\ D(z)&=1+(b_{20}+b_{21}-b_{10}b_{11})z^{-1}+b_{20}b_{21}z^{-2}\end{aligned} \quad (13)$$

This second-order MIMO model is also shown in Fig.2.

Remaining Structures

Let all subscripts in the following analysis range from 0 to N-1. From (8), and the *decimation theorem*[2],

$$y_k(n)=y(nN+k)=v(nN),\ v(n)=\sum_{\ell=-\infty}^{\infty}x(\ell)c_k(n+k-\ell) \quad (14)$$

$$V(z)=z^kX(z)C_k(z),\ Y_k(z)=\frac{1}{N}\sum_{n=-\infty}^{\infty}V(W^pz^{(1/N)}),\ W=e^{-j2\text{p}/N} \quad (15)$$

Thus $Y_k(z)=\dfrac{z^{(k/N)}}{N}\displaystyle\sum_{p=0}^{N-1}X(W^pz^{(1/N)})C_k(W^pz^{(1/N)})W^{pk}$ (16)

$$Y(z)=\sum_{k=0}^{N-1}z^{-k}Y_k(z^N)=\sum_{p=0}^{N-1}X(W^pz)\left[\frac{1}{N}\sum_{k=0}^{N-1}C_k(W^pz)W^{pk}\right] \quad (17)$$

Again from (8) and a generalisation of the *decimation theorem*[2]

$$y(n)=\sum_{\ell=-\infty}^{\infty}x(\ell)r_{<\ell>}(n-\ell)=\sum_{p=-\infty}^{\infty}\sum_{\ell=0}^{N-1}x(pN+\ell)r_\ell(n-pN-\ell) \quad (18)$$

$$\begin{aligned}\Rightarrow Y(z)&=\sum_{\ell=0}^{N-1}R_\ell(z)X_\ell(z^N)z^{-\ell}\\ &=\sum_{p=0}^{N-1}X(W^pz)\left[\frac{1}{N}\sum_{\ell=0}^{N-1}R_\ell(z)W^{p\ell}\right]\end{aligned} \quad (19)$$

And from (19)

$$\begin{aligned}Y_k(z)&=\frac{1}{N}\sum_{i=0}^{N-1}(W^iz^{(1/N)})^kY(W^iz^{(1/N)})\\ &=\frac{z^{(k/N)}}{N}\sum_{\ell=0}^{N-1}X_\ell(z)z^{(-\ell/N)}\sum_{i=0}^{N-1}R_\ell(W^iz^{(1/N)})W^{i(k-\ell)}\end{aligned} \quad (20)$$

Now eqns (18) and (19) suggest the *multiple-input model* in Fig.3, while eqns (14) and (17) suggest the *multiple-output model* in Fig.4.

From eqns (17) and (19) we get the two *modulator filter models (and implicit bifrequency map [2,13])* of Fig's 5 & 6, with

$$Y(z)=\sum_{p=0}^{N-1}X(W^pz)T_p(z) \quad (20a)$$

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[2] Let $x(n)\leftrightarrow X(z)$, then it is not difficult to show that: $\displaystyle\sum_{n=-\infty}^{\infty}x(nN+k)z^{-n}=\sum_{p=0}^{N-1}(W^pz^{\frac{1}{N}})^kX(W^pz^{\frac{1}{N}}),\ W=e^{-j2\text{p}/N}$.

$$\boxed{T_p(z) = \frac{1}{N}\sum_{\ell=0}^{N-1} R_\ell(z) W^{p\ell} = \frac{1}{N}\sum_{k=0}^{N-1} C_k(W^p z) W^{pk}} \quad (21)$$

From (10) and (11)

$$Y(z) = \sum_{k=0}^{N-1} z^{-k} Y_k(z^N) = \sum_{\ell=0}^{N-1}\left\{\sum_{r=0}^{N-1} z^{-r} H_{r\ell}(z^N)\right\} X_\ell(z^N) \quad (22)$$

and comparing (19) and (22)

$$\boxed{R_\ell(z) = z^\ell \sum_{r=0}^{N-1} H_{r\ell}(z^N) z^{-r}} \quad (23)$$

Now from (9)

$$\begin{aligned} C_k(z) &= \sum_{n=0}^{\infty} c_k(n) z^{-n} = \sum_{n=0}^{\infty} r_{<k-n>}(n) z^{-n} \\ &= \sum_{p=0}^{N-1}\left\{\sum_{n=0}^{\infty} r_{<k-p>}(nN+p) z^{-(nN+p)}\right\} \quad (23a) \\ &= \sum_{p=0}^{N-1} \frac{1}{N} \sum_{\ell=0}^{N-1} W^{\ell p} R_{<k-p>}(W^\ell z) \end{aligned}$$

Now substituting from (23) into (23a) then after some additional manipulation

$$\boxed{C_\ell(z) = z^{-\ell} \sum_{p=0}^{N-1} H_{\ell p}(z^N) z^p} \quad (24)$$

Now substituting (24) into (21)

$$\boxed{T_p(z) = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{r=0}^{N-1} z^{r-k} W^{pr} H_{kr}(z^N)} \quad (25)$$

From (23), and using a simple result[3]

$$z^{-k} R_k(z) = \sum_{r=0}^{N-1} H_{rk}(z^N) z^{-r}$$

$$\Rightarrow z^{-\ell} H_{\ell k}(z^N) = \frac{1}{N}\sum_{p=0}^{N-1} W^{p\ell} W^{-pk} z^{-k} R_k(W^p z)$$

$$\boxed{\Rightarrow H_{k\ell}(z) = \frac{z^{(k-\ell)/N}}{N} \sum_{p=0}^{N-1} W^{p(k-\ell)} R_\ell(W^p z^{(1/N)})} \quad (26)$$

Similarly from (24)

$$\boxed{H_{k\ell}(z) = \frac{z^{(k-\ell)/N}}{N} \sum_{p=0}^{N-1} W^{p(k-\ell)} C_k(W^p z^{(1/N)})} \quad (27)$$

Now substituting (27) and (26) into (23) and (24) respectively gives

$$\boxed{\begin{aligned} R_\ell(z) &= \frac{1}{N}\sum_{r=0}^{N-1}\sum_{p=0}^{N-1} W^{p(r-\ell)} C_r(W^p z) \\ C_\ell(z) &= \frac{1}{N}\sum_{r=0}^{N-1}\sum_{p=0}^{N-1} W^{p(\ell-r)} R_r(W^p z) \end{aligned}} \quad (28)$$

From (21), using these two expressions for $T_p(z)$ in (28) gives

$$\boxed{R_\ell(z) = \sum_{p=0}^{N-1} W^{-p\ell} T_p(z), \quad C_\ell(z) = \sum_{p=0}^{N-1} W^{-p\ell} T_p(zW^{-p})} \quad (29)$$

Finally, substituting for $R_k(W^p z^{(1/N)})$ (from (29) into (26)) gives:

$$\boxed{H_{k\ell}(z) = \frac{z^{(k-\ell)/N}}{N} \sum_{p=0}^{N-1}\sum_{r=0}^{N-1} W^{p(k-\ell)} W^{-r\ell} T_r(W^p z^{(1/N)})} \quad (30)$$

Now the inter-relationships between the five popular models for LPTV systems have been derived and are contained in the boxed equations. The nomenclature used is both self-contained, clear and consistent. *Note that the results presented can be obtained in a number of different ways.* For example, the general model of Fig.2 may be obtained by replacing each LTI $R_k(z)$ in Fig.3 with an equivalent structure from Fig.4.

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[3] $X(z) = \sum_{n=0}^{N-1} x(n) z^{-n} \Rightarrow x(n) z^{-\ell} = \frac{1}{N}\sum_{p=0}^{N-1} W^{p\ell} X(W^p z).$

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