FAST FRACTIONAL FOURIER TRANSFORM

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ABSTRACT

The *fractional Fourier transform* (FRFT) is a one-parametric generalization of the classical Fourier transform. Since it's introduction in $1980th$, the FRFT has been found a lot of applications and used widely nowadays in signal processing. Space and spatial frequency domains are the special cases of the fractional Fourier domains. They correspond to the 0th and 1st fractional Fourier domains, respectively. In this paper, we briefly introduce the multi-parametrical FRFT and its fast algorithm.

1 Introduction

Fourier analysis is one of the most frequently used tools in signal processing and in many other scientific disciplines. In the mathematical literature a generalization of the Fourier transform known as the *fractional Fourier transform* $\mathcal{F}^\alpha$ (FRFT), was proposed some years ago. It is known [3]-[6], that classical FFT is particular case of FRFT. Fourier space and spatial frequency domains are special cases of fractional Fourier domains. They correspond to the αth fractional Fourier domains ($\alpha = 0$ and $\alpha = 1$, respectively). In 1937, Condon wrote a paper called "Immersion of the Fourier transform in a continuous group of functional transformation" [1]. In 1961, Bargmann extended the FRFT in his paper [2], in which he gave definition of the FRFT, one based on Hermite polynomials as an integral transformation. If $H_n(\sqrt{2\pi}t)$ is a Hermite polynomial of order n then functions

$$\Psi_n(t) = \frac{2^{1/4}}{\sqrt{2^n n!}} H_n(\sqrt{2\pi}t)\exp(-\pi t^2)$$

for $n = 0, 1, 2, \ldots$ are eigenfunctions of the Fourier transform

$$\mathcal{F}[\Psi_n(t)] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Psi_n(t)e^{2\pi t\tau}dt = \lambda_n \Psi_n(t),$$

with $\lambda_n = i^n$ being the eigenvalue corresponding to the nth eigenfunction and form the orthonormal set of functions:

$$\int_{-\infty}^{+\infty} e^{-2\pi t^2}\frac{2^{1/4}}{2^n n!}H_n(\sqrt{2\pi}t)\frac{2^{1/4}}{2^m m!}H_m(\sqrt{2\pi}t)dt = \delta_{mn}. \quad (1)$$

According to Bargmann the FRFT $\mathcal{F}^\alpha := [\mathcal{F}^\alpha(\omega,t)]$ of order α may be defined through its eigenfunctions

$$\mathcal{F}^\alpha(\omega,t) := \sum_{n=0}^{\infty}\lambda_n^\alpha \Psi_n(\omega)\Psi_n(t) =$$

$$= \sqrt{2}e^{[-i\pi(\omega^2+t^2)]}\sum_{n=0}^{\infty} H_n(\sqrt{2\pi}\omega)H_n(\sqrt{2\pi}t),$$

where $\mathcal{F}^\alpha(\omega,t)$ is the kernel of the FRFT.

Obviously, the functions $\Psi_n(t)$ are eigenfunctions of FRFT

$$\mathcal{F}^\alpha[\Psi_n(t)] = \lambda_n^\alpha \Psi_n(t),$$

corresponding to the nth eigenvalues λ_n^α.

Of course, for $\alpha = 1$, $\mathcal{F}^1(\omega,t) = e^{i\omega t}$. If $0 < |a| < 2$ and $\alpha := 2\varphi/\pi$, then

$$\mathcal{F}^1(\omega,t) = \frac{\exp\left[-i\left(\frac{\pi \mathrm{sgn}(\sin\varphi)}{4} - \frac{\varphi}{2}\right)\right]}{\sqrt{\varphi}}\times$$

$$\times\exp\left[i\left(\pi\frac{\omega^2 - 2\omega t\cos\varphi + t^2}{\sin\varphi}\right)\right].$$

In 1980, Namias reinvented the FRFT again in his paper [3]. This approach was extended by McBride and Kerr [4]. The FRFT was restricted to pure mathematical purposes. Very few publications appeared. Then Mendlovic and Ozaktas introduced the FRFT into the field of optics [5] in 1993. Afterwards, Lohmann [6] reinvented the FRFT based on the Wigner-distribution function and opened the FRFT to bulk-optics applications. The Wigner-distribution of a function $f(t)$ is defined as

$$W_f(t,\omega) := \int f(t+\frac{\tau}{2})f^*(t+\frac{\tau}{2})\exp(-2i\pi\tau\omega)d\tau.$$

There is a following relationship between the fractional FRFT and Wigner-distribution a function $f(t)$:

$$W_{\mathcal{F}^\alpha[f]}(t,\omega) = W_f(t\ \cos\varphi - \omega\ \sin\varphi, t\ \sin\varphi + \omega\cos\varphi),$$

i.e. FRFT is a rotation operation applied over the Wigner plane. This relationship has been proposed as the definition of FRFT by Lohmann in [6].

In this paper we briefly introduce the multi-parametric FRFT and develop corresponding fast algorithm.

2 Multi-parametric fractional Fourier transform

Discrete Fourier transform (DFT) $\mathcal{F}$ of the lenght N is defined by

$$F(k) := \frac{1}{N}\sum_{n=0}^{N-1} f(n)e^{\frac{2\pi}{N}nk},$$

where $f(n)$ is the signal of the lenght N from the signal vector space $\mathbf{V}_N(e_0, e_1, \ldots, e_{N-1})$, spaned on natural basis $e_0, e_1, \ldots, e_{N-1}$. In operator notation we write $\mathbf{F} = \mathcal{F}\mathbf{f}$. DFT has characteristic equation $\lambda^4 = 1$ since $\mathcal{F}^4 = I$, where I is identity operator. Consequently, the DFT $\mathcal{F}$ has only four eigenvalues in the form of solutions equations $\lambda^4 = 1 : \lambda(k) = e^{j\frac{2\pi}{4}k}$, $k = 0, 1, 2, 3$. If $N = 2^n$, then this eigenvalues has multiplicities $2^{n-2}+1$, $2^{n-2}-1$, 2^{n-2}, $2^n - 2$, respectively.

The Hermite polynomials $H_n(\sqrt{2\pi}t)$ (but not $\Psi_n(t)$) form a set that is orthonormal with respect to the weight function

$$w(t) = \exp(-2\pi t^2) = \exp\left(-\frac{t}{\sqrt{\frac{1}{2\pi}}}\right)^2.$$

It is well known that the discrete counterpart of a Gaussian window is a binomial window, i.e.

$$w(i) = \frac{1}{2^N}C_N^i$$

for $i = 0, 1, \ldots, N$. The (discrete) orthonormal polynomials that are associated with this window are known as Krawtchouk's polynomials

$$K_n(i) = \sum_{k=0}^{n}(-1)^{n-k}C_{N-i}^{n-k}C_i^k$$

for $i, n = 0, 1, \ldots, N$, i.e.

$$\sum_{i=0}^{n} C_N^i\left[\frac{1}{\sqrt{2^N C_N^n}}K_n(i)\right]\left[\frac{1}{\sqrt{2^N C_N^m}}K_m(i)\right] = \delta_{nm}.$$

The functions $\psi_n(i) := \sqrt{\frac{C_N^i}{2^N C_N^n}K_n(i)}$ form the set of the eigenvectors of DFT:

$$\mathcal{F}[\psi_n(i)] = \lambda_n\psi_n(t).$$

For large values of N, the binomial window reduces to a Gaussian window. More specifically,

$$\lim_{N\to\infty}\frac{1}{2^N}C_N^{t+(N/2)} =$$

$$= \frac{1}{\sqrt{\pi N/2}}\exp\left[-\left(\frac{t}{\sqrt{N/2}}\right)^2\right]$$

for $t = -(N/2), \ldots, N/2$. It can be shown that the same limiting process turns a Krawtchouk polynomial into a Hermite polynomial, i.e.

$$\lim_{N\to\infty} K_n\left(t + \frac{N}{2}\right) = \frac{1}{\sqrt{2^n n!}}H_n\left(\frac{t}{\sqrt{N/2}}\right)$$

Hence, the discrete Hermite transform of lenght N approximates the analog Hermite transform of spread $\sigma = \sqrt{N/2}$.

Let $\mathbf{U} = [\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_{N-1}]$ be the matrix of eigenvectors of an discrete Fourier transform $\mathcal{F}$, then

$$\mathbf{U}\mathcal{F}\mathbf{U}^{-1} = \mathrm{diag}\{\lambda(k)\}$$

and

$$\mathcal{F} = \mathbf{U}^{-1}\mathrm{diag}\{\lambda(k)\}\mathbf{U}.$$

Definition 1 Let $\alpha_0, \alpha_1, \ldots, \alpha_{N-1}$ be arbitrary real numbers from $[0, 1]$, then

$$\mathcal{F}^{\alpha_0,\alpha_1,\ldots,\alpha_{N-1}} :=$$

$$= \mathbf{U}^{-1}\left\{\mathrm{diag}(\lambda_0^{\alpha_0}(k), \lambda_1^{\alpha_1}(k), \ldots, \lambda_{N-1}^{\alpha_{N-1}}(k))\right\}\mathbf{U} \quad (2)$$

is called *the multi-parametric fractional Fourier transform* (MFRFT).

The set of all multi-parametric fractional $\mathcal{F}$-transforms form Abelian group $(\mathbf{R}/4)\times(\mathbf{R}/4)\times\ldots\times(\mathbf{R}/4)$, since

$$\mathcal{F}^{\alpha_0,\alpha_1,\ldots,\alpha_{N-1}}\mathcal{F}^{\beta_0,\beta_1,\ldots,\beta_{N-1}} =$$

$$= \mathcal{F}^{\alpha_0\oplus\beta_0,\alpha_1\oplus\beta_1,\ldots,\alpha_{N-1}\oplus\beta_{N-1}},$$

where $\oplus$ is the symbol of addition modulo 1. If $\alpha_i = \alpha$, $\forall i = 0, 1, \ldots, N-1$, then $\mathcal{F}^{\alpha_0,\alpha_1,\ldots,\alpha_{N-1}} = \mathcal{F}^{\alpha}$ is the classical fractional Fourier transform.

According to definition 1 efficient calculation of (1) require fast computational algorithm for transformation by $\mathbf{U}$ matrix ($\mathbf{U}$-transform).

3 Fast U-Transform for DFT

Let $\mathbf{Rot}\Big[\varphi_m \mid k_m, l_m\Big] := \begin{bmatrix} \cos\varphi_m & \sin\varphi_m \\ -\sin\varphi_m & \cos\varphi_m \end{bmatrix}_{k_m,l_m}$ by elementary Jacobi-Givens rotation in 2-D coordinate plane (e_{k_m}, e_{k_l}) of the signal space $\mathbf{V}(e_0, e_1, ..., e_{N-1})$. In this paper we use sequential method for reduction of the classical Fourier transform using finite sequence of elementary Jacobi-Givens rotations:

$$\mathcal{F}_{(m)} :=$$

$$= \mathbf{Rot}\Big[+\varphi_m \mid k_m, l_m\Big]\cdot\mathcal{F}_{(m-1)}\cdot\mathbf{Rot}\Big[-\varphi_m \mid k_m, l_m\Big],$$

where $m = 0, 1, ..., S$, $\mathcal{F}_{(0)} := \mathcal{F}$, $\mathcal{F}_{(S)} = \mathrm{diag}\{\lambda(k)\}$. The angles φ_m are determined so that $w^{(m)}_{k_m,l_m} = w^{(m)}_{l_m,k_m} = 0$, where $\mathcal{F}_{(m)} := \left[w^{(m)}_{k_m,l_m}\right]$.

Matrix $\mathcal{F}$ (without the first column and first row) are centro-symmetric (persymmetric or double symmetric). Therefore it is block-diagonalized by $\frac{N}{2} - 1$ rotations of matrix $\mathbf{X}^{\oplus}_{0,N} :=$

$$\left(\sqrt{2}\mathbf{Rot}\left[0^o \mid 0, \frac{N}{2}\right] \prod_{i=1}^{\frac{N}{2}-1} \frac{2}{\sqrt{2}}\mathbf{Rot}\left[\frac{\pi}{4} \mid i, N-i\right]\right) :$$

$$\mathcal{F}_{(\frac{N}{2}-1)} = \mathbf{X}^{\oplus}_{0,N}\mathcal{F}\mathbf{X}^{\oplus}_{0,N} = \Delta\mathbf{X}^{\oplus}_{0,N}\left[\mathbf{C}^{\oplus}_{0,N} \oplus \overline{I}\mathbf{S}\overline{I}\right]\mathbf{X}^{\oplus}_{0,N},$$

where Δ is diagonal matrix, consisting of only $+1$ and -1, $\mathbf{C}_{\frac{N}{2}+1}$, $\mathbf{S}_{\frac{N}{2}-1}$ are discrete cosine and sine transforms, respectively, and $\overline{I}$ is the antidiagonal matrix.

Transforms $\mathbf{C}_{\frac{N}{2}+1}$ and $\mathbf{S}_{\frac{N}{2}-1}$ are centro-symmeric and they are block-diagonalized by $\frac{N}{2} - 1$ rotations of matrix $\mathbf{X}^{\oplus}_{1,\frac{N}{2}-1} = \mathbf{X}^{\oplus}_{0,\frac{N}{4}+1} \oplus \mathbf{X}^{\oplus}_{1,\frac{N}{4}-1}$. After $N-2$ rotations we obtain the matrix

$$\mathcal{F}_{(N-2)} = \mathbf{X}^{\oplus}_{1,\frac{N}{2}-1}\mathcal{F}_{(\frac{N}{2}-1)}\mathbf{X}^{\oplus}_{1,\frac{N}{2}-1},$$

which is reducible to block-diagonal form.

Example 1 Let $N = 8$ then

$$\mathcal{F}_{(3)} = \mathbf{X}^{\oplus}_{(0,8)}\mathcal{F}\mathbf{X}^{\oplus}_{(0,8)} =$$

$$= \frac{\Delta}{\sqrt{8}}\left(\begin{bmatrix} 1 & \sqrt{2} & \sqrt{2} & \sqrt{2} & 1 \\ \sqrt{2} & \sqrt{2} & \cdot & -\sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \cdot & -2 & \cdot & \sqrt{2} \\ \sqrt{2} & -\sqrt{2} & \cdot & \sqrt{2} & -\sqrt{2} \\ 1 & -\sqrt{2} & \sqrt{2} & -\sqrt{2} & 1 \end{bmatrix} \oplus\right.$$

$$\left.\oplus \begin{bmatrix} -\sqrt{2} & 2 & -\sqrt{2} \\ 2 & 2 & -2 \\ -\sqrt{2} & -2 & -\sqrt{2} \end{bmatrix}\right)$$

and

$$\mathcal{F}_{(6)} = \mathbf{X}^{\oplus}_{(1,8)}\mathcal{F}\mathbf{X}^{\oplus}_{(1,8)} =$$

$$= \frac{\Delta}{\sqrt{8}}\left(\begin{bmatrix} 2 & \cdot & 2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -2\sqrt{2} \\ 2 & \cdot & -2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \sqrt{2} & \cdot \\ \cdot & -2\sqrt{2} & \cdot & \cdot & \cdot \end{bmatrix} \oplus\right.$$

$$\left.\begin{bmatrix} -2\sqrt{2} & \cdot & \cdot \\ \cdot & \cdot & -2\sqrt{2} \\ \cdot & -2\sqrt{2} & \cdot \end{bmatrix}\right).$$

If

$$\mathbf{T}^3_8 := \left[I_1 \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus I_3\right]$$

is the permatation matrix, then

$$\mathcal{F}'_{(6)} = \mathbf{T}^3_8\mathcal{F}_{(6)}\mathbf{T}^3_8 =$$

$$= \frac{\Delta}{\sqrt{8}}\left[\begin{bmatrix} 2 & 2 \\ 2 & -2 \end{bmatrix} \oplus \begin{bmatrix} 0 & -2\sqrt{2} \\ -2\sqrt{2} & 0 \end{bmatrix} \oplus\right.$$

$$\left.\oplus 2\sqrt{2} \oplus (-2\sqrt{2}) \oplus \begin{bmatrix} 0 & -\sqrt{2} \\ -2\sqrt{8} & 0 \end{bmatrix}\right].$$

Thus, we have to do three rotation into planes (e_1, e_2), (e_3, e_4), (e_7, e_8), in order to obtain scalar-diagonal matrix $\mathcal{F}_9 \equiv \mathrm{diag}(1, -1, 1, -1, -j, j, -j)$. These rotations are $\mathbf{T}^4_8 =$

$$= \frac{\sqrt{2}}{2}\left[\sqrt{2}\begin{bmatrix} c^1_3 & -s^1_3 \\ s^1_3 & c^1_3 \end{bmatrix} \oplus \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \oplus\right.$$

$$\left.\oplus\sqrt{2} \oplus \sqrt{\oplus}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right]$$

Therefore

$$\mathbf{U} := \mathbf{T}^4_8\mathbf{T}^3_8\mathbf{X}^{\oplus}_{(1,8)}\mathbf{X}^{\oplus}_{(0,8)} =$$

$$= \begin{bmatrix} C^1_3 & -S^1_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ S^1_3 & C^1_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & +d & +d & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & +d & -d & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +d & +d \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +d & -d \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} +1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & +1 & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \sqrt{2} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & +1 & \cdot & +1 & \cdot & \cdot & \cdot & \cdot \\ +1 & \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \sqrt{2} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot & 1 \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} \sqrt{2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & +1 & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & +1 & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \sqrt{2} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & +1 & \cdot & +1 & \cdot & \cdot \\ \cdot & \cdot & +1 & \cdot & \cdot & \cdot & +1 & \cdot \\ \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & +1 \end{bmatrix},$$

where $C^k_m := \cos(\frac{k\pi}{2^m})$, $S^k_m := \sin(\frac{k\pi}{2^m})$, $d = \frac{\sqrt{2}}{2}$.

Finally, as an example, we give the matrix representation of fast transform for $N = 16$:

$$\mathbf{U} := \mathbf{T}^5_{16}\mathbf{T}^4_{16}\mathbf{T}^3_{16}\mathbf{X}^{\oplus}_{(1,16)}\mathbf{X}^{\oplus}_{(0,16)},$$

where

$$\mathbf{T}^5_{16} =$$

$$= \begin{bmatrix} +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & +d & +d & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -d & +d & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & C^1_3 & \cdot & S^1_3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & C^1_3 & \cdot & S^1_3 & \cdot & \cdot \\ \cdot & \cdot & \cdot & -S^1_3 & \cdot & C^1_3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -S^1_3 & \cdot & C^1_3 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 \end{bmatrix} \oplus$$

$$\oplus \begin{bmatrix} +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & C^1_3 & \cdot & S^1_3 & \cdot & \cdot \\ \cdot & \cdot & \cdot & C^1_3 & \cdot & S^1_3 & \cdot \\ \cdot & \cdot & -S^1_3 & \cdot & C^1_3 & \cdot & \cdot \\ \cdot & \cdot & \cdot & -S^1_3 & \cdot & C^1_3 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 \end{bmatrix},$$

$$\mathbf{T}_{16}^{4} =$$

$$= \begin{bmatrix} +d & \cdot & -d & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ +d & \cdot & +d & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & +d & \cdot & \cdot & +d & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & +d & -d & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & +d & +d & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -d & \cdot & \cdot & +d & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & C_4^1 & S_4^1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -S_4^1 & C_4^1 \end{bmatrix} \oplus$$

$$\oplus \begin{bmatrix} C_4^3 & -S_4^3 & \cdot & \cdot & \cdot & \cdot & \cdot \\ S_4^3 & C_4^3 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & +d & \cdot & \cdot & +d & \cdot \\ \cdot & \cdot & \cdot & +d & -d & \cdot & \cdot \\ \cdot & \cdot & \cdot & +d & +d & \cdot & \cdot \\ \cdot & \cdot & -d & \cdot & \cdot & +d & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 \end{bmatrix},$$

$$\mathbf{T}_{16}^{3} =$$

$$= \begin{bmatrix} +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot \end{bmatrix} \oplus$$

$$\oplus \begin{bmatrix} +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot \end{bmatrix},$$

$$\mathbf{X}_{(1,16)}^{\oplus} =$$

$$= \begin{bmatrix} +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & +1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & +1 & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \sqrt{2} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & +1 & \cdot & +1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & +1 & \cdot & \cdot & \cdot & +1 & \cdot & \cdot \\ \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot \\ +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 \end{bmatrix} \oplus$$

$$\oplus \begin{bmatrix} +1 & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & +1 & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & +1 & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \sqrt{2} & \cdot & \cdot & \cdot \\ \cdot & \cdot & +1t & \cdot & +1 & \cdot & \cdot \\ \cdot & +1 & \cdot & \cdot & \cdot & +1 & \cdot \\ +1 & \cdot & \cdot & \cdot & \cdot & \cdot & +1 \end{bmatrix},$$

$$\mathbf{X}_{(0,16)}^{\oplus} =$$

$$= \begin{bmatrix} \sqrt{2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \sqrt{2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot \\ \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot \\ \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 \end{bmatrix}$$

Similar expressions were found for **U**-transforms of lengths until 256.

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