A TIME-FREQUENCY METHOD FOR NONLINEAR SYSTEM CLASSIFICATION IN PRESENCE OF NOISE

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ABSTRACT

This paper presents a method for the classification of nonlinear systems through the study of the free oscillations in the time-frequency plane, when the measured data are affected by noise. Nonconservative SDOF (Single Degree Of Freedom) oscillators described by a nonlinear second order differential equation are considered. The nonlinearity is due to a nonlinear function of the state variable, which produces free oscillations with a time-variant spectrum. The method used for the classification is a substantial modification of a basic algorithm proposed by the same authors for noise-free data. In presence of noise improved performances are obtained with the new algorithm.

1 INTRODUCTION

This paper deals with the problem of classifying nonlinear systems described by a nonlinear differential equation

$$\ddot{u}(t) + c\dot{u}(t) + f(u(t)) = 0 \tag{1}$$

where c is a small positive parameter, called *damping factor*, and $f(u)$ is a function of the state variable u identifying the type of nonlinearity. The class of systems described by (1) is called *SDOF* (Single Degree Of Freedom), because a single one-dimensional state variable is present in the differential equation, and *nonconservative*, because of the positive damping c, which causes the signal $u(t)$ to decay. A classical SDOF nonconservative system is the nonlinear oscillator.

Equation (1) is used as a fundamental model in many areas of science, and the study of its properties constitutes a subject of undoubted interest. In particular in identification problems (for example in structural diagnostic), the interest is often in the classification of the type of nonlinearity. In literature this problem is studied with methods based on the analytic signal [3], inverse Volterra-Wiener series [8], Higher Order Spectra (HOS) [7]. In [4] a completely new approach has been proposed, based on the identification of the nonlinear function $f(u)$ (from now on, called *the nonlinearity*) through the signal $u(t)$ represented in the Time-Frequency plane. In this plane, as explained in Sect. 2.1, the nature of the *free oscillations* generated by the system, set up with initial conditions not too far from equilibrium (weak nonlinearity), can be easily observed. In fact, as it is well known from the theory of the nonlinear oscillations [6], the signal $u(t)$ produced in this situation is generally composed of a time-variant fundamental frequency plus a set of time-variant higher harmonics. The time evolution and the energy of both fundamental and harmonics depend on the type of nonlinearity.

The method proposed in [4] is very efficient when data are noise-free. In this paper a modified algorithm is proposed, able to overcome the performance degradation due to the presence of noise in the measured data.

Simulation results prove the validity of the method.

2 NONLINEAR OSCILLATORS

The nonlinearity in (1) is due to the function $f(u)$. Only when $f(u) = ku$ the system becomes linear, and (1) becomes the representative equation of the well known harmonic oscillator. In this case an exponentially damped sinusoid, with frequency f_0, is produced by the system, when set up with some initial values $u(0)$, $\dot{u}(0)$. The generated signal is often referred to as free oscillation or oscillation, with frequency f_0 independent from the amplitude of the oscillation.

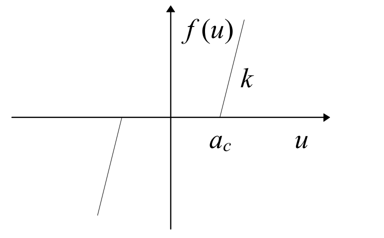
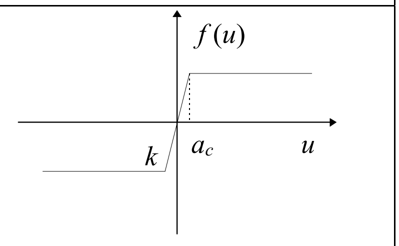
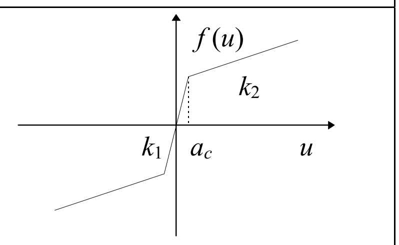
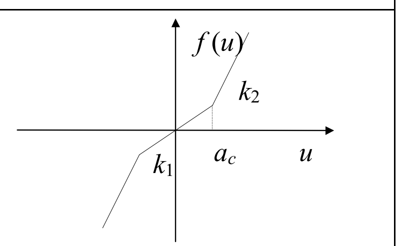
In a nonlinear oscillator the frequency of the oscillation depends on the signal amplitude, and therefore it can vary according to the amplitude variations. Moreover the signal $u(t)$ looses its exact sinusoidal behavior, and it will be composed by a fundamental frequency plus a set of higher harmonics.

The nonlinearities considered in this paper are indicated in Table 1. They are particularly interesting in structural diagnostic, where they describe the typical nonlinear behavior of beams under dynamic loading.

2.1 Time-Frequency Representation of the Free Oscillations

The Time-Frequency Distributions (TFD) are a powerful tool for the representation and the analysis of signals with time-varying spectra, [2]. Each TFD is defined as a two-dimensional transformation (from the *Time* do-

Table 1: Nonlinearity Class

Type	Definition
A	
B	
C	
D	
E	$f(u) = \alpha u + \beta u^3$

main to the *Time-Frequency* domain); a TFD, applied to a generic signal $x(t)$, produces a two-dimensional function $D_x(t, f)$, defined in a plane, known as TF (Time-Frequency) plane.

The time variation of both the fundamental and the higher harmonics of the free oscillations are well represented in the TF plane. For what concerns the method proposed in this paper, the key point of this signal representation is the fact that all the useful information for the algorithm can be extracted from the TF plane.

2.2 Instantaneous Frequency

A crucial point of the analysis based on TFDs is the possibility of associating an *instantaneous frequency* $f_i(t)$ to a generic signal $x(t)$, defined as the average frequency of $x(t)$ at a particular time. It is well known [2], that the instantaneous frequency obtained with the Wigner Distribution (WD) exhibits a "physical" meaning in terms of harmonic decomposition of a signal $x(t)$. For this reason the Wigner distribution will be used for characterising the time variation of the fundamental in our application.

3 DESCRIPTION OF THE CLASSIFICATION METHOD

The classification method proposed in this paper is based (as the basic algorithm in [4]) on the assumption that the time variation of the fundamental, denoted as $f_0(t)$, may be a discriminating element among different nonlinearities. The method is performed in three steps.

Estimation in the TF Plane. An estimate $\hat{f}_0(t)$ of the fundamental frequency, based on the instantaneous frequency introduced in Sect. 2.2, is first accomplished. In particular the extraction of the fundamental is performed by using a variant of a method called *WD (Wigner Distribution) peak detection algorithm* [1]. The variant has been designed to take into account the presence of noise.

First a masking process in the TF plane is performed, to cut off the noisy data around the fundamental frequency. The mask is designed to well fit the time-varying behavior of the fundamental. Second the maximum of the masked Wigner is evaluated for each time t, leading to the estimate $\hat{f}_0(t)$.

Analytical Approximation. The second step of the method consists in producing an alternative expression $\tilde{f}_0(t)$ of the fundamental related to the system parameters. Two approximations are necessary to obtain this expression.

- *Conservative Approximation.* The nonconservative system described by (1), in the case of a weak nonlinearity, can be considered as locally conservative ($c = 0$), and then described by the equation

$$\ddot{u}(t) + f(u(t)) = 0 \qquad (2)$$

valid in a short time interval. The oscillation $u(t)$ will be periodic, but not sinusoidal [6]. Therefore it will be composed by a fundamental frequency and a number of harmonics. Because of the nonlinearity the fundamental frequency will be a function of its amplitude a.

An analytical approximation $\tilde{f}_{0,\mathrm{A}}$ of the fundamental can be obtained from (2) through the use of techniques, known in nonlinear analysis, as *Perturbation Methods* [5]. The approximations for the nonlinearities of Table 1 can be found in [4].

In general $\tilde{f}_{0,\mathrm{A}}$ depends on the parameters $p_1, p_2, ..., p_n$ of the nonlinear function $f(u)$, and on the oscillation amplitude a. Therefore it can be written as $\tilde{f}_{0,\mathrm{A}}(a; p_1, p_2, ..., p_n)$.

It is important to emphasize the fact that this expression is valid only locally. In the actual systems $c \neq 0$, so causing the fundamental amplitude to decay. The idea proposed in [4] is to assume the approximation $\tilde{f}_{0,\mathrm{A}}(a; p_1, \ldots, p_n)$ valid also in the case $c \neq 0$, by substituting the constant a with a time function $a(t)$ representing the time variation of the fundamental amplitude. Therefore $\tilde{f}_{0,\mathrm{A}}(a; p_1, p_2, ..., p_n)$ will be written as $\tilde{f}_{0,\mathrm{A}}(a(t); p_1, p_2, ..., p_n)$. The simulation results shown in [4] validate this assumption for measured noise-free data.

- *Low-Energy Harmonics Approximation.* The fundamental amplitude $a(t)$ is estimated as the total oscillation amplitude. Its value, denoted as $\hat{a}(t)$, is evaluated with sophisticated techniques of signal synthesis in the TF plane. This part is innovative with respect to the basic algorithm in [4], and represents the key point of the method when data are affected by noise. The idea is to construct a synthetic signal $s(t)$, whose instantaneous frequency is forced to be $\hat{f}_0(t)$, and with exponential decaying amplitude

$$s(t) = a(t) cos\hat{\varphi}_0(t) = \beta e^{-\alpha t} cos\hat{\varphi}_0(t) \tag{3}$$

where

$$\hat{\varphi}_0(t) = \int_0^t 2\pi \hat{f}_0(t')dt' \tag{4}$$

Then a minimization problem in the least square sense is set up, between the Wigner distribution of the noisy signal and $s(t)$

$$\min_{\alpha,\beta} \|W[s] - W[u]\|_2^2 \tag{5}$$

The parameters $\hat{\alpha}, \hat{\beta}$ of the minimum point determine the estimated fundamental amplitude

$$\hat{a}(t) = \hat{\beta} e^{-\hat{\alpha}(t)} \tag{6}$$

Combining these two approximations, the expression of the fundamental is finally reached:

$$\begin{aligned} \tilde{f}_0(t) &= \tilde{f}_0(t; p_1, p_2, ..., p_n) \\ &= \tilde{f}_{0,\mathrm{A}}(\hat{a}(t); p_1, p_2, ..., p_n) \end{aligned}$$

Minimization Step. The third step of the method is based on a matching algorithm between $\tilde{f}_0(t; p_1, p_2, \ldots, p_n)$ and $\hat{f}_0(t)$. The hypothesis is made that the nonlinearity responsible of a given measured signal $u(t)$ should be able to best fit its fundamental $\hat{f}_0(t)$. For each nonlinearity the minimum M_i of $\|\tilde{f}_0(t; p_1, p_2, \ldots, p_n) - \hat{f}_0(t)\|$ is evaluated for each nonlinearity. In our example $i \in [1, 5]$ is an index for the five nonlinearities of Table 1. Nonlinearities are then classified with respect to the distance M_i reached in this data fitting problem. The identified nonlinearity is the one that reaches the minimum M_i.

Notice that this method is also able to assign a set of estimated parameters $(\hat{p}_1, \hat{p}_2, \cdots, \hat{p}_n)$ to M_i. Therefore both classification and parameters estimation are attained with the proposed algorithm.

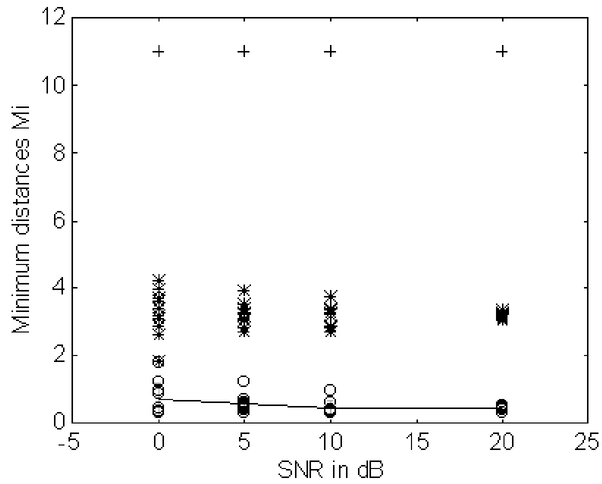


Figure 1: Results of the classification process. Circles ("O"): Minimum distances reached by approximation type C; Stars ("*"): Type B; Pluses ("+"): Type A, D, E. The continuous line links the mean value of the type C distances for every SNR.

4 VALIDATION OF THE METHOD

The method has been validated by simulation for the five nonlinearities in Table 1. Five test signals $u_I(t)$ (I=A, B, C, D, E) were created by simulation, each one representing the free oscillation generated by a non linear system of type I. The results of the five tests, shown in [4], validate the method in the ideal case of noise-free data.

In case of noisy data similar results have been obtained. A signal $u(t)$ has been generated with the nonlinearity type C of Table 1. Then a gaussian noise has been added to this signal, with SNR values of 20, 10, 5, 0 dB. Ten noisy signals have been generated for each SNR value.

The method has been applied to this set of signals, and the results are shown in Fig. 1. The circles ("O") represent the minimum reached by the approximation function type C, the stars ("*") indicate the minimum reached for type B, while the pluses ("+") refer to the other three nonlinearities. It is possible to see that all the circles are located in the bottom of the diagram (there is only one quite ambiguous value for 0 dB), and this means that the approximation function type C always reaches the minimum distance with respect to the others. The identified nonlinearity is hence always type C, the correct one, proving the validity of the method.

5 CONCLUSION

In this paper an original method for the classification of nonlinear systems has been presented, based on the time-frequency analysis of the free oscillations. The method is robust, as it works also in presence of high-level noise.

References

[1] B. Boashash, *Time-frequency signal analysis: methods and applications*. Melbourne: Longman Cheshire, 1992

[2] L. Cohen, *Time-frequency analysis*. Englewood Cliffs, NJ: Prentice Hall, 1995

[3] M. Feldman and S. Braun, "Identification of nonlinear system parameters via the instantaneous frequency: application of the Hilbert transform and Wigner Ville techniques," in *Proc. 13th International IMAE*, 1995, pp. 637-642.

[4] L. Galleani, L. Lo Presti, A. De Stefano, "A Method for Nonlinear System Classification in the Time-Frequency Plane," submitted to *Signal Processing*

[5] A. H. Nayfeh, *Perturbation methods*. New York: John Wiley & Sons, 1973

[6] A. H. Nayfeh, *Nonlinear oscillations*. New York: John Wiley & Sons, 1979

[7] C. L. Nikias and A. Petropulu *Higher-Order Spectral Analysis: A Nonlinear Signal Processing Framework*. Englewood Cliffs, NJ: Prentice Hall, 1993

[8] M. Schetzen, *The Volterra and Wiener theories of nonlinear systems*. New York: John Wiley & Sons, 1980